Optimization of Stochastic Response Surfaces Subject to Constraints with Linear Programming

THESIS

Robert Garrison Harvey, First Lieutenant, USAF

AFIT/GOR/ENS/92M-14

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY
AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

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STUDENT: Robert Garrison Harvey, 1Lt, USAF | CLASS: GOR-92M

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COMMITTEE:	NAME/DEPARTMENT	SIGNATURE
Advisor	Maj Kenneth Bauer	Kenneth W Bauer
Reader	Lt Col Joseph Litko	Joseph R. Litko

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Robert Garrison Harvey, B.S.

First Lieutenant, USAF

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Preface

The purpose of this study was to investigate the optimization of stochastic response surfaces with linear programs. It was found that using the traditional approach of estimating a response surface and using it as the objective function of a linear program yielded a bias in the mean solution. Also, nonoptimal extreme points have a large probability of being chosen. This research investigated a method to overcome these disadvantages and obtain an improved solution.

Table of Contents

List of Figures

List of Tables

Abstract

This research investigated an alternative to the traditional approaches of optimizing a stochastic response surface subject to constraints. This research investigated the bias in the expected value of the solution, possible alternative decision variable settings, and a method to improve the solution. A three step process is presented to evaluate stochastic response surfaces subject to constraints. Step 1 is to use a traditional approach to estimate the response surface and a covariance matrix through regression. Step 2 samples the objective function of the linear program (i.e., response surface) and identifies the extreme points visited. Step 3 presents a method to estimate the optimal extreme point and present that information to a decision maker.

1 Introduction

1.1 Background

Constrained and unconstrained optimization is a diverse and growing field with applications in many areas. Two key optimization areas of interest in this research were: optimization of simulation output with constraints and stochastic programming.

Response surface methodology can be used to optimize a system modeled by a simulation. Response surface methodology typically incorporates three areas: experimental design, regression analysis, and optimization. The response surface (i.e., the fitted regression model) generated from a stochastic simulation is a metamodel that approximates the system being simulated. The coefficients of the response surface representing the simulation in a design region are random variables. The random, or stochastic, nature of the coefficients in the response surface is at the core of this research. Typically, after establishing and validating a response surface the analyst ignores its stochastic nature and employs it to estimate optimum operational conditions in a resource constrained environment (11:138).

Stochastic programming, usually a completely separate field from response surface methodology, concentrates on the stochastic nature of elements in math programming problems. One aspect of stochastic programming concentrates on the random nature of the coefficients in the objective function in a linear program while assuming the constraints are deterministic. A literature review suggests there is little incorporation of stochastic programming to simulation optimization; the research done is limited in scope and solves only individual problems. No research effort has been found that investigates the general process. Davis and West observe that:

...recent research has demonstrated that there is indeed a role for the employment of mathematical programming procedures with simulations... in modeling situations where uncertainties are small, mathematical programming would be the preferred choice. However, many decisionmaking problems do not meet this criterion. For example, long-term production/inventory planning is often analyzed using a mathematical programming formulation even though this problem is known for its uncertainties arising in long-term demand forecasts, in the costs of input material, and in manufacturing productivities... Another classic problem for which considerable uncertainties often exist is the investment portfolio selection problem... For any problem with severe uncertainties the adopted approach of merging simulation with mathematical programming can be applied. (5:200,209).

1.2 Problem Statement

The goals of this research are threefold. First, this research will characterize the impact of the stochastic nature of the response surface on the optimization of a simulation with multiple constraints. Second, this research will develop a means to identify the "true" optimum point in practice. Third, this research will develop a method to increase confidence in the point estimate on the optimal answer to the constrained optimization problem.

1.3 Objectives & Related Methods

The goals of this research was met by accomplishing the following objectives.

Objective 1: To characterize the impact of the stochastic nature of the objective function (response surface) coefficients on the distribution of Z^* (the optimal solution to the linear programming problem) and identify both the basis and extreme point changes.

Methods:

1. Generated a variety of linear programming problems to investigate the topic.
2. Wrote a computer program that performs a Monte Carlo process on the linear program by varying the objective function coefficients and error term to be

sampled. Collected data on the distribution of Z^*, basis changes, and sampled optimum extreme points.

3. Defined a variety of variance-covariance matrices ($\sigma^2(X'X)^{-1}$) to generate the multivariate normal distribution of the coefficients.

4. Defined five noise levels to test.

5. Ran a Monte Carlo procedure and evaluate the mean and variance for each extreme point, basis, and for the overall solution.

<u>Objective 2:</u> Investigate a method to design an experiment on the response surface coefficients that samples the response surface in an effective way.
<u>Method:</u> Used a Box-Behnken design and investigated modifications to the design that better sampled the response surface.

<u>Objective 3:</u> Evaluated ways to better estimate the "true" optimal extreme point, and present the data, to a decision maker, in an efficient way.
<u>Methods:</u>

1. Used a ranking and selection procedure to screen extreme points and identify the "best" extreme point.

2. Investigated presentation of data through a histogram.

2 Literature Review

2.1 Introduction

The following paragraphs will review literature pertinent to optimization of a stochastic response surface subject to multiple constraints. Specifically, the discussion covers the topics of experimental design, constrained simulation optimization, and stochastic programming with Monte Carlo simulation.

2.2 Discussion

2.2.1 Experimental Design

Much research has been done on experimental designs since the 1950s (7:571). The basic goal of experimental design is to choose the settings of the factor levels in a set of experiments. In simulation, experimental design is the process of developing a scheme to conduct an experiment on a simulation and collect output information deriving the maximum amount of useful information with a minimum expenditure of resources. In experimental design jargon, the input variables to the simulation are factors and the levels (values) those variables can take are treatments.

Experimental design in simulation is distinguished from experimental design in general by two things. First, in simulation the analyst specifies the factors and treatments at the beginning to get the optimal design. Second, the input random variables for each simulation experiment are controlled by the analyst who can exploit this by making comparisons between experiments more precise, such as using the same pseudo-random number stream for different experiments (i.e., using common random numbers).

Typical objectives of experimental design in a simulation are to

1. Understand the effect of the factors on the experimental output.
2. Estimate the parameters of interest.

3. Make a selection from a set of alternatives.

4. Find the treatment levels for all factors that produce the optimum response (3:105-223).

There are many types of designs, from the simple 2^n factorial to specialty designs to measure lack of fit, fit second order models, or control variance. The list of experimental designs is extensive, although "the central composite [design] is used more than any other family of RSM designs" (11:147).

A key influence in the stability in predicted variance is the rotatability of the design: while a high degree of rotatability is desirable, a perfectly rotatability design is not needed (11:138). Rotatability refers to the way variance propagates through the design; in a perfectly rotatable design variance is only a function of the distance from the center of the design. In other words, in a perfectly rotatable design, variance can be defined as increasing in concentric circles expanding out from the center of the design. In a near-rotatable design the variance can be thought of in the same way except the circles are not perfect, but slightly elliptical. In light of variance considerations, Myers, Khuri and Carter conclude

> ...the RSM user needs to learn from the Taguchi approach that system variability should be a major component in the analysis. A similar argument can be made for consideration of the distribution of variance of the prediction in the assessment of experimental designs. Often the success of the RSM endeavor is dependent on the properties of y [hat] at different locations in the design space. Many standard designs have prediction variances which increase dramatically as one gets close to the design parameter. As a result, any conclusions drawn (regarding choice of optimal or improvements in operating conditions) concerning response near the design boundary are suspect. Yet we see very little that deals with this in design assessment or comparisons among designs. We too often evaluate a design on the basis of one number (say, D-efficiency) when the important aspects of behavior are multidimensional. (11:152)

Biles and Swain observe "the n-dimensional simplex design, which employs n+1 design points at the vertices of a regular simplex, gives the greatest efficiency in terms of information per design point " (2:138). Experimental design is a diverse area; only through investigation will identification of the family of "best" designs occur.

2.2.2 Constrained Simulation Optimization

Classical simulation optimization is a broad field. Typical approaches to optimization include heuristic searches, complete enumeration, random searches, steepest ascent, coordinate searches, pattern searches and many others (6:117-121).

Biles and Swain present several strategies for constrained simulation optimization that appear to be representative of the main constrained simulation optimization efforts going on today. They fit and validate a response surface using an n-dimensional simplex, biradial, or equiradial design. They account for the variance of the error term, but they assume the "response surfaces are the expected values of the observed responses." (2:135). They do not account for the stochastic nature of the response surface, but employ a recursive method by applying an optimization procedure and then returning to the simulation model until the optimal criteria are met. Their procedures include direct search techniques, first-order response surface, and second-order response surface procedures. The type of constraints Biles and Swain used are either simple (upper and lower bounds) or multiple (e.g., budget or resource) constraints (2:135-137).

The choice between point estimation verses interval estimation, while important to the analyst, has only recently been addressed:

> Many users of RSM allow conclusions to be drawn concerning the nature of a response surface and the location of optimal response without taking into account the distributional properties of the estimated attributes of the underlying response surface. Although the distribution of these quantities has not been considered directly, efforts have been made to develop interval estimates. Box and Hunter (1954) used a version of Fieller's theorem to develop a 100(1 - alpha)% confidence region for the location of the stationary point. The construction of a confidence interval around the response at the stationary point of the true surface has only recently been a subject of interest in the statistical literature. Khuri and Conlon (1981) gave an expression for the bounds of an interval conditional on the estimated location of the stationary point. (11:146)

The experimental design impacts the distributional form of the underlying data, which in turn, "has an impact on the estimation of the model parameters and on the inferences drawn from an RSM analysis" (11:146).

Constrained optimization of stochastic simulations through linear programming is starting to appear in the literature, but no literature has been found that evaluates the impact of the stochastic nature of the response surface in the general case. Myers, Khuri and Carter support this: "Nearly all practical RSM problems are truly multiple response in nature. Sophisticated ways of solving [stochastic] multiple-response problems are not generally well known, however" (11:147).

2.2.3 Stochastic Programming

Stochastic programming considers three stochastic areas of math programming: the objective function $\mathbf{c}^T\mathbf{x}$, constraint matrix $\mathbf{A}$, and the right hand side $\mathbf{b}$ vector. Traditionally, stochastic programming has not been incorporated in simulation optimization analysis.

There are three classical approaches to solving stochastic programming problems where the coefficients vary: expected value, "Fat," and "Slack" approaches. The expected value approach uses the point estimate for the coefficients to solve the math programming problem. The "Fat" approach chooses a pessimistic value for the coefficients in the math programming problem. The "Slack" solution method assumes randomness in the constraint matrix and the right hand side and adds a penalty function to the objective function. The "Slack" method is a two-stage problem that assumes the decisionmaker can adjust a previous decision (9:463-470). These three approaches to stochastic programming are used when the coefficients are either random or constrained to a given set. Another area of stochastic programming is chance constrained optimization: "[In this] approach, it is not required that the constraints should always hold, but we shall be satisfied if they hold in a given proportion of cases or, to put it differently, if they hold with given probabilities" (13:75).

Most analytical methods assume no basis changes take place, and many do not account for changes in the extreme point solutions (12:211). Under these assumptions analytical methods have been developed to estimate the mean and the variance of the stochastic program:

> ...[Consider] the cost of minimizing $C = c^T x$ [the objective function], where the components of the vector c had a joint normal distribution with means m_j and covariance matrix V. C is then also normally distributed with mean $M = m^T x$ and variance $S^2 = x^T V x$.
>
> If all coefficients and constants of the constraints are fixed, and if we want to minimize the expected value of C, then the problem is reduced to the deterministic program of minimizing $m^T x$. But let us assume that we want to minimize the expected utility of C, which we define as: 1 - exp(-aC) where a is a positive constant, which economists call a measure of the aversion to risk. (13:23-24)

Vajda goes through a proof that reduces the problem to a "deterministic program of minimizing $aM - .5a^2S^2$, a function which is quadratic in x" (13:24).

The feasibility of the solution to a stochastic program is a topic of much research. Randomness in the constraint matrix or the right hand side **b** vector can cause the solution to be super-optimal and infeasible. If only the objective function is stochastic, the solution will remain feasible and only the optimal value will change (13:3).

Bard and Chatterjee introduce a perspective of the variability of a design that is specifically concerning objective function bounds for the inexact linear programming problem with generalized cost coefficients. They conclude that increasing the "oblongness" of the variance of the coefficients in the objective function rather than decreasing their volun.e provides better results when solving the stochastic linear program (1:491). In other words, uniform variance in the coefficients of the objective function is not necessarily desirable, but decreasing variance of one coefficient at the expense of others may lead to improved LP solutions.

Bracken and Soland present a paper on "a statistical decision analysis of a one-stage linear programming problem with deterministic constraints and stochastic criterion function" (4:205). This paper presents analytical and Monte Carlo methods to find the expected value of both perfect and sampled information, but it is not possible to solve the analytical problem when the objective function coefficients come from a multivariate normal distribution. In addition, this article presents a method to describe the "...distribution of the optimal value of the linear programming problem with stochastic objection function and [discusses] Monte Carlo and numerical integration procedures for estimating [the optimal value] distribution" (4:205).

A misconception exists that there is no need to evaluate the stochastic nature of the objective function in a linear program because the analyst can use sensitivity analysis to conduct a proper evaluation. Davis and West address this misconception:

> Post-optimal or sensitivity analysis provides the modeler with the ability to analyze the functional behavior of the optimal solution as parameter assignments are modified, but these methods again provide little insight toward the probable values that the optimal solution [of a stochastic problem] will assume. For this reason it is often difficult to choose a robust solution. (5:199)

Bracken and Soland observe that the optimal solution to any linear program will always occur at an extreme point: therefore only extreme points need to be considered. In other words, if all the extreme points can be identified, then the actual linear programming problem does not need to be solved, but instead a Monte Carlo procedure can be used to evaluate the probability distribution of the simulation output at each extreme point. The value of the extreme points can then be evaluated and compared to identify the top-ranking alternatives. This article refers to an article by C. E. Clark in which he describes a method (using a reduced set of extreme points) approximating the characteristics of the distribution of the "...maximum value of the (reduced) linear programming problem admitting only the p selected alternatives." (4:212,214). "p is a subset of the overall (assumed independent) extreme points.

> There are several reasons for studying the reduced linear programming problem. First of all, the number r of extreme points which comprise the set S may be so large that the decision maker finds it undesirable to consider all of them in his analysis of the decision situation. He may therefore decide to limit his further analysis to the p selected alternatives. The smaller number of alternatives thus available makes computation of the EVPI [Expected value of perfect information] and/or the EVSI [Expected value of sampled information] much more feasible. Second, the expected value of the maximum value of the reduced problem is a lower bound to the corresponding quantity for the original problem, and could therefore serve as an estimate of the expected value of the maximum value of the original problem. We would expect the bound to be best when the p selected alternatives are the p top-ranking alternatives (in order) with respect to the prior distribution on **c** [the unknown mean vector of a multivariate stochastic process]. (4:212)

Clark's procedure, as contrasted with a Monte Carlo simulation

> has the advantage that the results may be obtained quickly and cheaply for different selections of the alternative extreme points and/or different parameters in the distribution of **c**. The accuracy achieved with Clark's procedure is somewhat limited, however, especially when the distribution of **v** [where **v** is the objective function] is degenerate, whereas great accuracy can be achieved in Monte Carlo simulation if a sufficiently large number of draws is used...One difficulty is that the values obtained with Clark's procedure are dependent upon the order in which the variables are listed. (4:214-224)

Davis and West present both a decision theory approach and a Monte Carlo approach to solving this problem.

> Decision theory begins by predefining the alternative solutions that will be considered. Next potential realizations for the decisionmaking environment must be specified with *a priori* assignment of the probability that each realization occur. Using this information, the trade-offs among the alternatives are then analyzed, and the apparent optimal solution alternative is selected. Decision theory does generate the probability that each alternative will be the optimal solution. However, the analysis is limited by the number of predefined alternative solutions selected and the accuracy in the specification of the *a priori* probabilities for the potential states of the systems... generation of probabilistic bounds upon the optimum solution requires considerable computational effort with the mathematical programming approach whereas decision-theory approaches provide this information directly...Although these analyses are still typically

limited to the consideration of a finite set of alternatives, the methods of response surface methodology coupled with classical optimization approaches have been used to optimally assign key parameter values within simulated systems operating under the selected alternative. (5:199-200)

Davis and West, instead of defining the possible solution states, use the simulated decisionmaking scenarios from the Monte Carlo analysis and assume each scenario has an equal probability of occurring (5:207). The above approach belies the difficulty in identifying the possible scenarios and their probability of occurring. The paper was introduced as a contrast between the methods of decision-theory and Monte Carlo simulation, but the decision analysis builds on the results from the Monte Carlo analysis. Also, the paper presents an analysis of only a single study and not an evaluation of the overall process (5:199-209).

2.3 Summary

Incorporating stochastic programming in the optimization procedure for stochastic simulations with constraints has recently been used in limited cases, but no overall evaluation of the process has been done[1]. In solving problems with random coefficients

> difficulties arise from two sources. First, meaningful simulation models must be generated from which the uncertain parameter values can be sampled. Second, the ability to statistically analyze the results from numerous sample optimal solutions generated during the simulation must be demonstrated. Even if both difficulties are addressed, the task still remains of selecting the decision which provides the best compromise between optimality and risk. (5:209)

Morben, in solving a "real world" problem, demonstrates a case where using the expected value of a stochastic objective function leads to an answer which falls outside a 95% confidence bound found through a Monte Carlo analysis (10:27). This case clearly

[1] This research will analyize the overall process and look into ways to incorporate experimental design in the stochastic programming subportion.

demonstrates there is a risk in some situations if only the expected value is used, and it makes the case for incorporating some form of stochastic analysis. Experimental design, constrained simulation optimization, and stochastic programming arc all well developed fields while the need to incorporate aspects of all three seems to exist little research has been done to achieve this goal.

3 Phase I: Impact of Estimation Errors Methodology

3.1 Introduction

Phase I investigated how errors in estimating the response surface affect the estimate of the optimal solution of the maximizing linear program. Figure 1. shows the basic approach

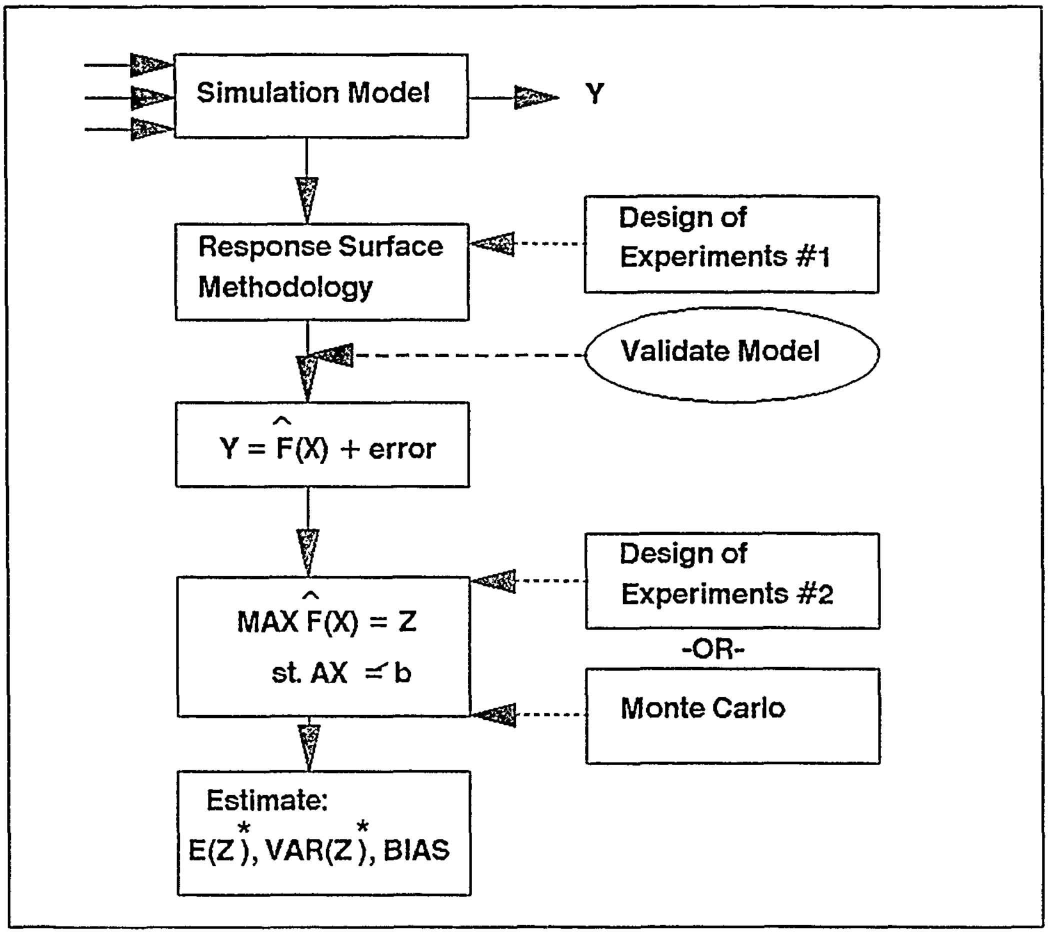


Figure 1. Analysis Flow

3.2 Starting Hypothesis

This research phase considers the simulation a black box that consists of a "Truth Model" plus noise.

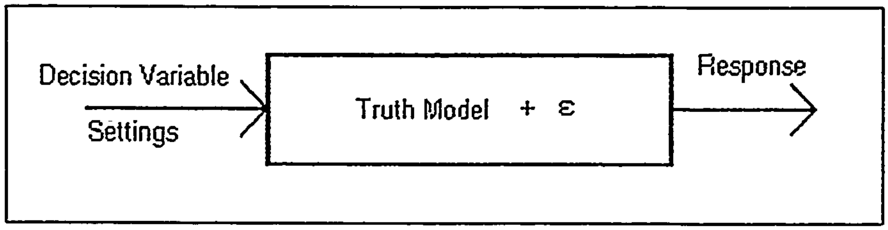


Figure 2. Black Box Simulation Model

The simulation response serves as input to the estimate of a response surface (the objective function of a linear program). The Functions:

$$Z^* = LP(\mathbf{C}, \mathbf{A}, \mathbf{b}) \tag{1}$$

$$\hat{Z}^* = LP(\hat{\mathbf{C}}, \mathbf{A}, \mathbf{b}) \tag{2}$$

define the optimal value Z^* (or estimated optimal value $\hat{Z}^*$) of a linear program employing the revised simplex method. Where

$\mathbf{C} = \mathbf{c}^T\mathbf{x}$ true (or known) objective function

$\hat{\mathbf{C}} = \hat{\mathbf{c}}^T\mathbf{x}$ estimated objective function

$\mathbf{A}$ = constraint matrix

$\mathbf{b}$ = right hand side vector.

$\mathbf{c}$ = true surface coefficients underlying the metamodel

$\hat{\mathbf{c}} = \mathbf{c} + \varepsilon$ estimated coefficients of objective function (response surface)

Assumption: $\varepsilon \sim N(0,\sigma^2(X^TX)^{-1})$

Phase I started with the premise that Z^* (the "true" optimum) is equal to the expected value of the parameters of the linear program:

$$Z^* = LP(E(\hat{\mathbf{C}}, \mathbf{A}, \mathbf{b})) \tag{3}$$

but, it will be demonstrated that Z^* is not, in general, equal to the expected value of the linear program with the estimated value of the objective function:

$$Z^* = LP(\mathbf{C}, \mathbf{A}, \mathbf{b}) = LP(E(\hat{\mathbf{C}}, \mathbf{A}, \mathbf{b})) \neq E(LP(\hat{\mathbf{C}}, \mathbf{A}, \mathbf{b})) = E(\hat{Z}^*) \tag{4}$$

Further, as the standard error in the estimates of the coefficients increases the bias in $\hat{Z}^*$ and $\sigma^2(\hat{Z}^*)$ increases.

3.3 Testing Hypothesis

This research used a computer program to test the hypothesis in Equation 4. The computer program uses a Monte Carlo approach; sampling from a "Truth Model" with noise it generates a response surface which is used as the objective function of a linear program. The objective function is then sampled using the variance-covariance matrix generated from the regression of the design matrix and response while collecting statistics at all stages (see Figure 3).

3.4 Investigating Indicated v. True

Besides characterizing the distribution of the estimated optimal solution, this research investigated what solution an analyst might expect in practice versus the "true" solution. The analyst estimates two key elements: the optimal extreme point ($\hat{EP}^*$) and the optimal value given that point ($\hat{Z}^*$). Theoretically, any changes to the linear program could change characteristics of the comparison, but this research will investigate whether there is a common trend to be identified.

To characterize the possible results the computer program generates an estimated objective function (response surface) and solves the linear program. Both the estimated optimal value and the estimated optimal extreme point are generated and compared to the true optimal extreme point, its value, and the value of the true function at the estimated extreme point (later these values are sorted and plotted to give a visual representation of the comparison). Contrasting these plots with plots of different linear programs and at different noise levels gives insights to the problem.

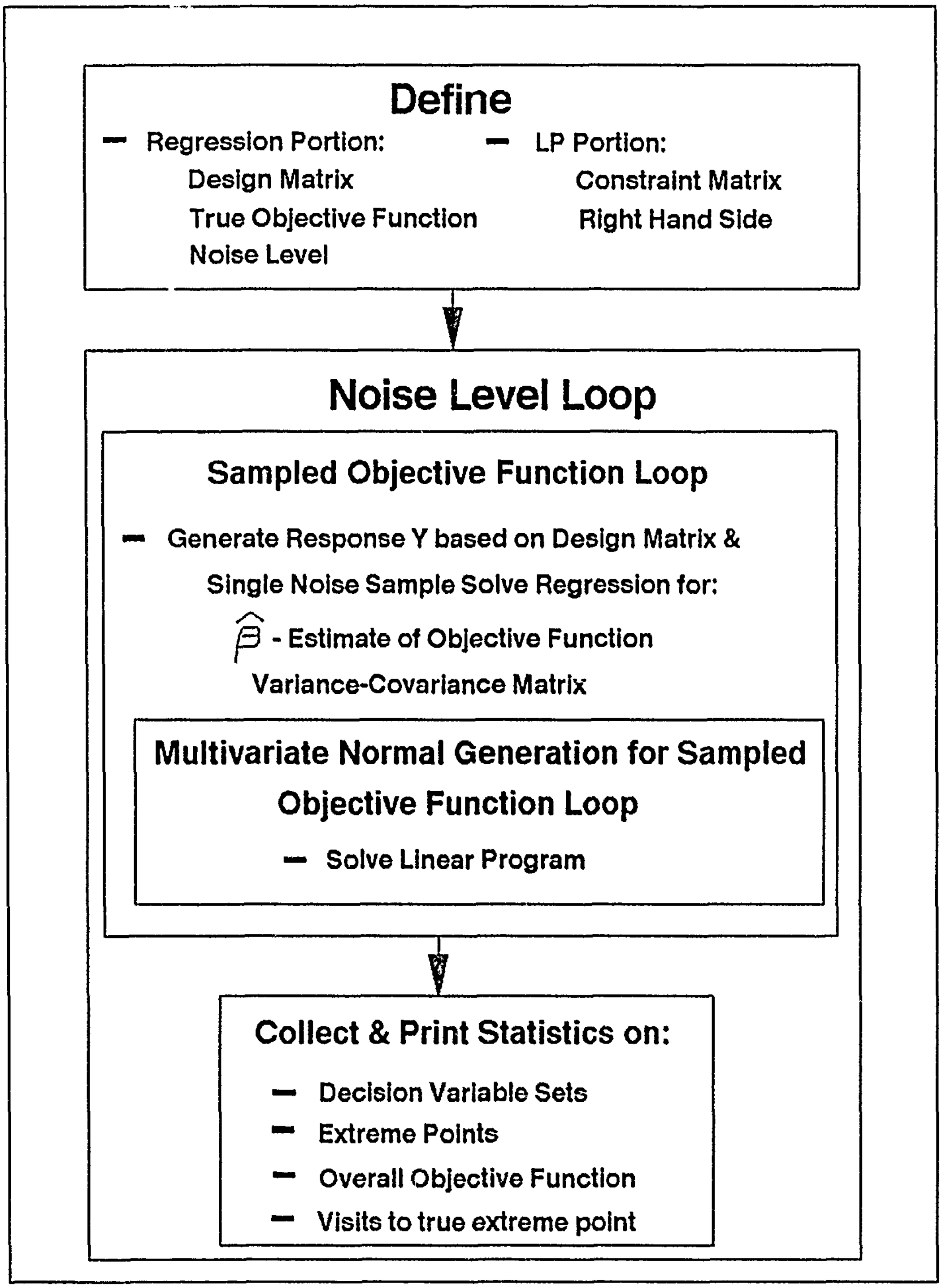


Figure 3. Computer Flow

4 Results Phase I

4.1 Introduction

There is no singular "true" answer to Phase I, but it appears the general process can be characterized. Consistent results were found for all maximization problems studied by evaluating different linear programs at different noise levels and generating random "Truth Models". The following results capture what can be expected in the general case. As noise is introduced in the estimation of the objective function the estimated optimal extreme point will vary as shown in Figure 4.

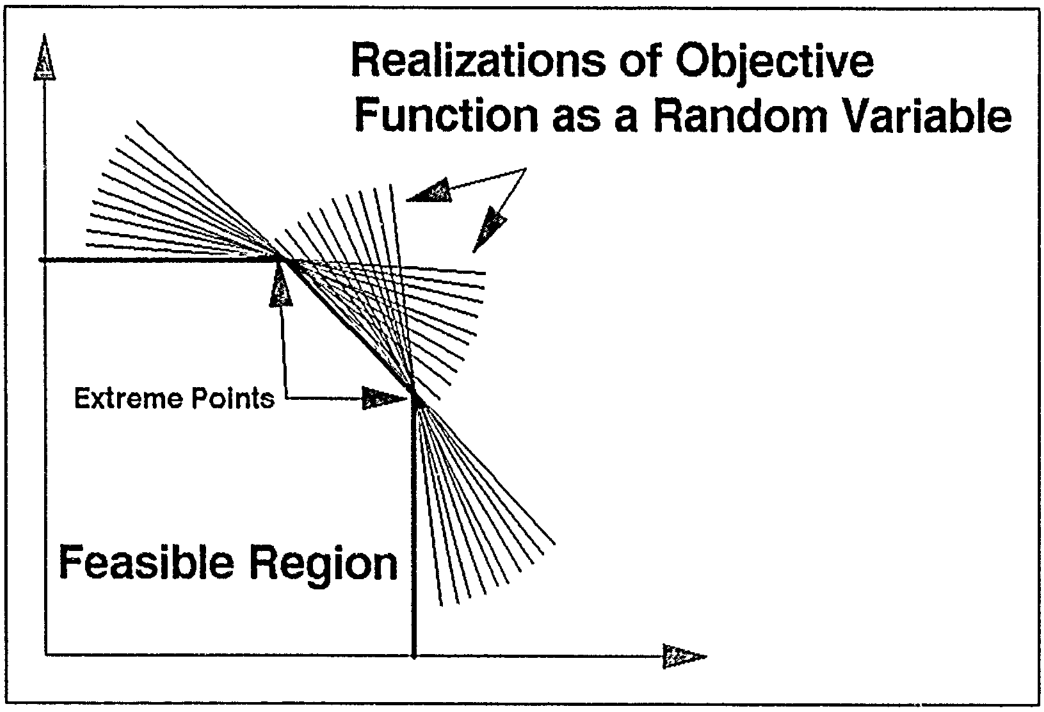


Figure 4. Objective Function Noise

4.2 Bias and Variance

An additional module was added to the computer program to generate random "truth models," constraint matrices, and right hand side vectors. The output from this module characterized this basic approach over a variety of linear programming problems.

Always, for the randomly generated maximization problems tested, the computer program indicated that the value of the "True" linear program is less than or equal to the value of the linear program with noise :

$$Z^* < E(LP(\hat{C}, \mathbf{A}, \mathbf{b})) = E(\hat{Z}^*) \tag{5}$$

This can be graphically represented in the following figure:

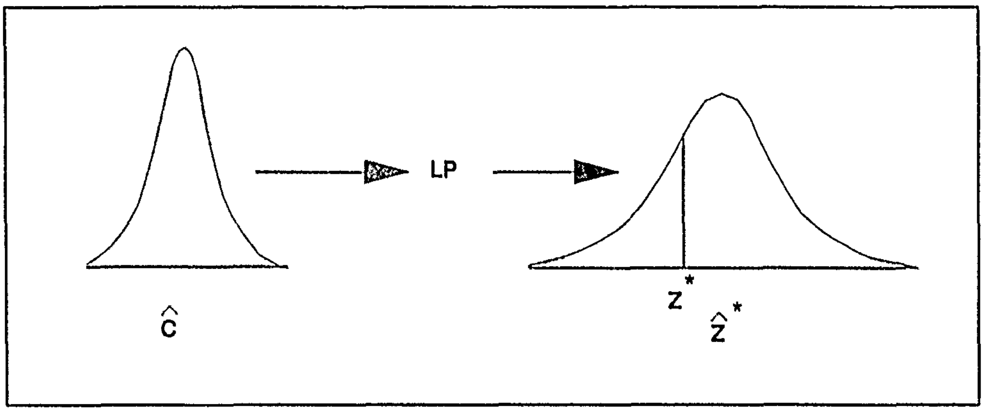


Figure 5. Noise Impact & Bias

this figure shows how a normally distributed estimate of the objective function $\hat{C}$ affects the normal distribution in the estimate of the optimal value $\hat{Z}^*$.

The high bias in the estimate of the mean was present in all linear programming problems analyzed (and those randomly generated). Also, as the noise level increased in a given problem the bias increased in a roughly linear trend; here the bias is the mean estimated optimum minus the true optimum.

$$\text{Bias} = E(\hat{Z}^*) - Z^* \tag{6}$$

The following figures illustrate a typical case where the standard error = σ(parameter estimates). The underlying linear program, in this case, is:

Maximize	$15x_1 +$	$17x_2 +$	$18x_3 +$	$20x_4 +$	10	
Subject to	$x_1 +$	$x_2 +$	$2x_3 +$	x_4	$\leq$	12
	$2x_1 +$	$x_2 +$	$-x_3 +$	x_4	$\leq$	14
	$-x_1 +$	$x_2 +$	$x_3 +$	$2x_4$	$\leq$	10
	x_1	x_2	x_3	x_4	$\geq$	0

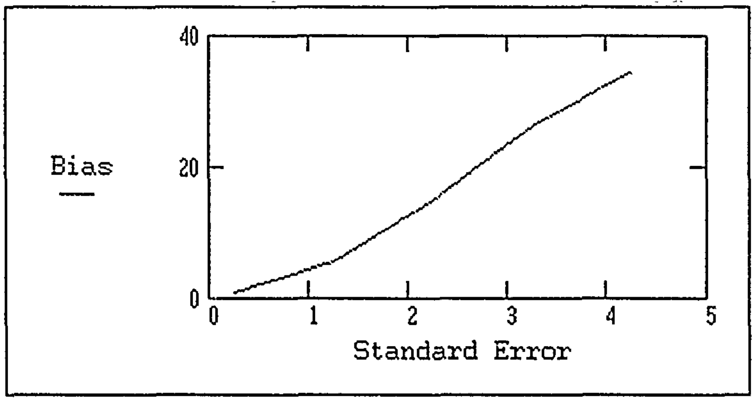


Figure 6. Bias Inflation

The bias increases as the standard error increases, and the standard deviation of $\hat{Z}^*$ follows a similar trend as shown in the Figure 7.

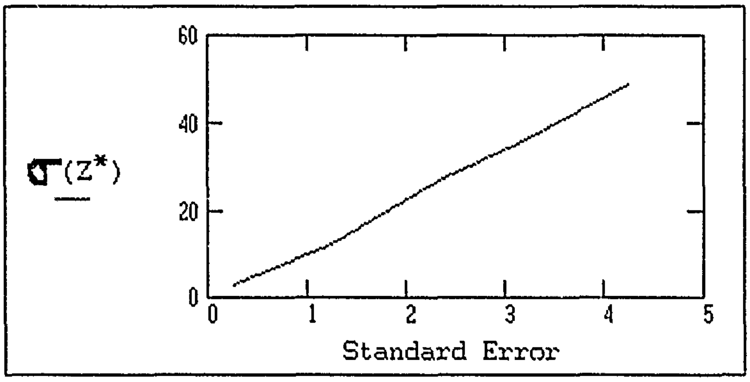


Figure 7. $\sigma(Z^*)$ Inflation

Combining the last two points, one can expect as σ(parameter estimates) increases not only does the bias increase, but the spread in $\hat{Z}^*$ increases.

4.3 Indicated v. Actual Solutions

The bias in the estimated value of the optimal answer only illustrates the trend over multiple realizations, but an analyst may be interested in what he can expect in one realization of the process. Here, let's go a step beyond the statistics in the previous section dealing with bias and analyze the contrast between each $\hat{Z}^*$ and the actual value of choosing that extreme point. In the following figures

True = value of "Truth Model" given the extreme point corresponding to $\hat{Z}^*$

Indicated = $\hat{Z}^*$

The extreme point for $\hat{Z}^*$ is that extreme point the linear program chooses as optimal. The following figures are characteristic of all problems evaluated. The following points are of interest:

1. The True solution is sorted in descending order where the first (left most) extreme point represents the "true" optimal extreme point.
2. The Indicated solutions are sorted in descending order around their extreme point.
3. All points in the plots are equally likely in a single realization of the process (8000 samples).
4. For a point estimate there is no way to know if the estimate is high or low without prior knowledge.
5. The distribution of the solution around each extreme point is normally distributed with a bias.
6. As σ increases the percent of visits to the "true" optimal extreme point decreases.
7. As σ increases there is a greater chance that the chosen extreme point will be greatly inferior to the "true" optimal extreme point, but there is still a chance $\hat{Z}^*$ with be much higher than even the "true" optimal extreme point Z^*.
8. The chance of selecting the solution with the chosen extreme point within 10% using $\hat{Z}^*$ occurs only about 10% of the time.
9. Extreme point changes occur where a True value step change occurs and where the Indicated value jumps from a low value to a high value.

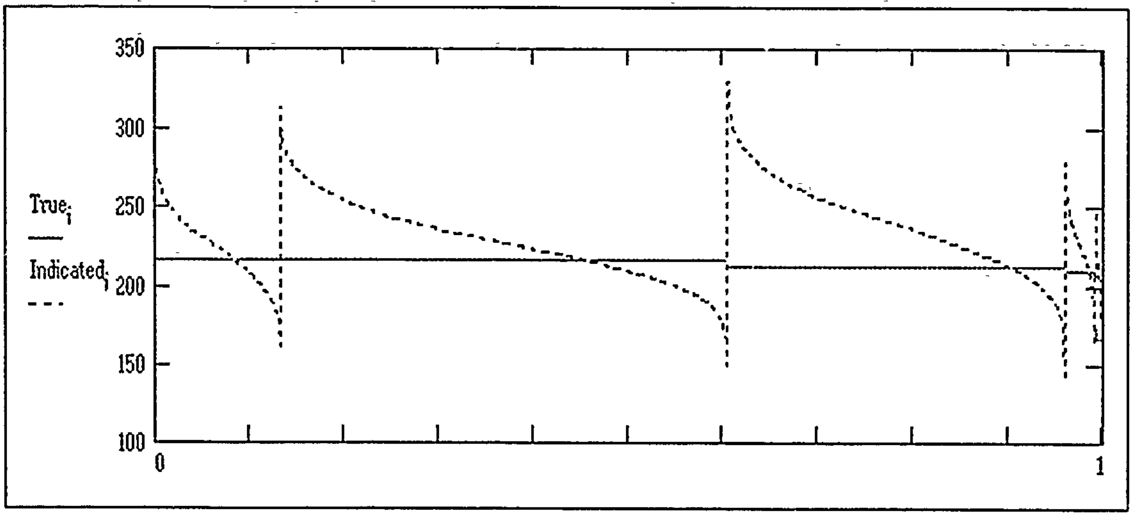


Figure 8 Indicated Z^* vs Actual $\sigma = 2.25$

When $\sigma = 2.25$ visits to the "true" optimal extreme point occur about 14% of the time, and about 98% of the solutions are "close" to the "true" optimal extreme point.

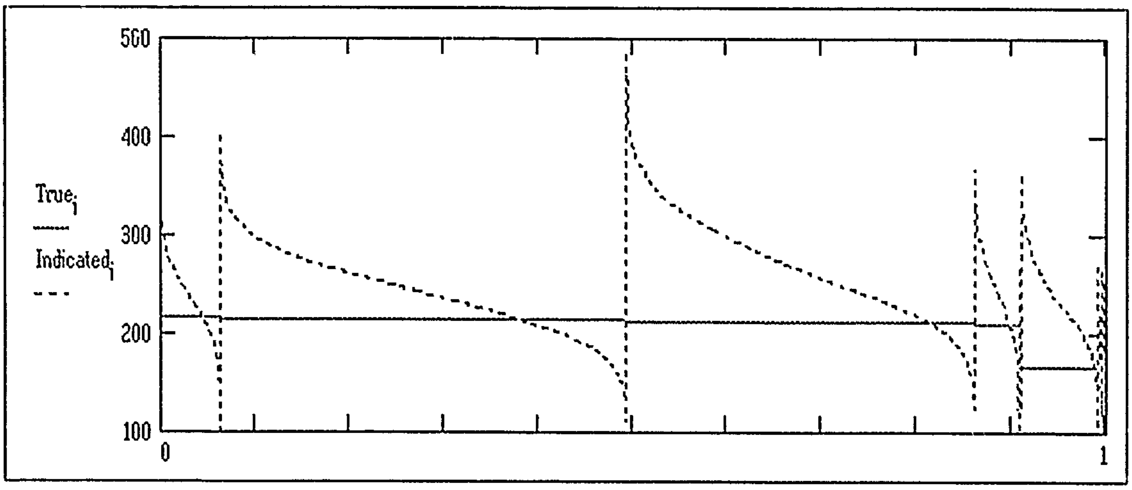


Figure 9 Indicated Z^* vs Actual $\sigma = 3.25$

When $\sigma = 3.25$ selecting the "true" optimal extreme point occurs about 7% of the time, and about 92% of the solutions are "close" to the "true" optimal extreme point.

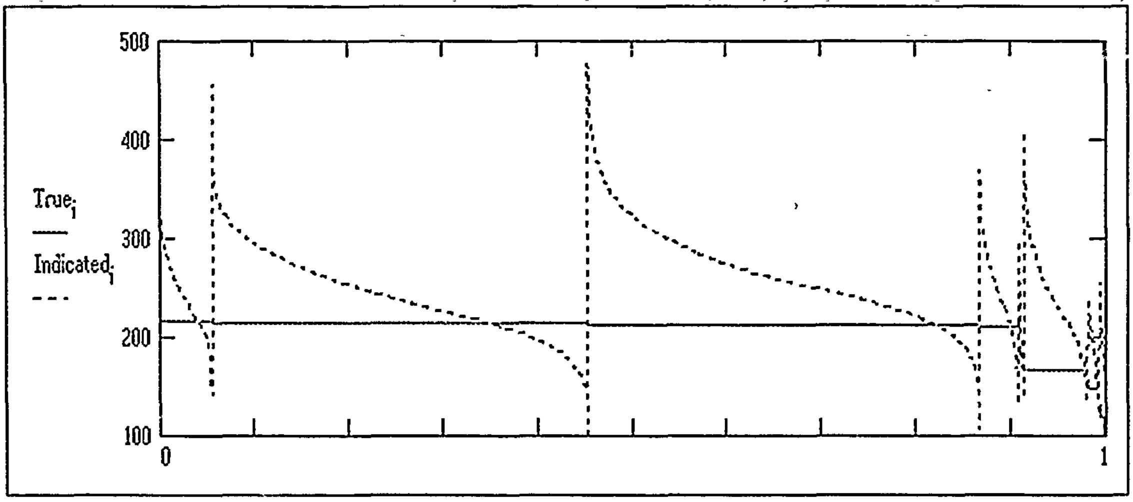


Figure 10 Indicated Z^* vs Actual $\sigma = 4.25$

When $\sigma = 4.25$ selecting the "true" optimal extreme point occurs about 6% of the time, and about 91% of the solutions are "close" to the "true" optimal extreme point.

A single realization of the process (or point estimate) gives little information on how the chosen extreme point will actually perform. Other techniques must be investigated to put this information into practical application.

5 Phase II Sampling Extreme Points Methodology

5.1 Introduction

Phase I illustrated that limited information about the true solution can be obtained from a single realization of the process. The estimated extreme point may lead to a highly biased solution compared to the true extreme point. Phase II investigated how to obtain the true extreme point using two methods. The first method samples the generated objective function (in a Monte Carlo fashion) using the variance-covariance matrix from the regression analysis and catalogs the extreme points visited. The second method samples the generated objective function via a design and catalogs the extreme points visited. Identifying the optimal extreme point may be possible by sampling the simulation at each extreme point visited--this will be investigated in Phase III. Also, empolying a screening technique will be investigated to improve the efficiency of the process.

5.2 Starting Hypothesis

As a starting hypothesis, this research assumes that given an initial response surface and its associated variance-covariance matrix we can sample the "true" optimal extreme point.

Pr("true" optimal EP sampled | initial estimated EP & $\hat{\sigma}^2(X^TX)^{-1}$) ≈ 1

as $N \rightarrow \infty$

where

EP = extreme point

initial estimated EP = extreme point identified when the objective function is assumed to be deterministic and the LP is solved once.

N = number of objective function samples.

$\hat{\sigma}^2(X^TX)^{-1}$ = variance-covariance matrix driving multivariate normal sampling.

5.3 Visits to true Extreme Points via Sampling

In practice, analysts usually only have one estimate of the objective function, but in this research the computer program can generate any number of objective functions. Calculating a variance-covariance matrix for each response surface the program then uses it as input to generate a multivariate normal sample. The computer program then samples from this distribution an arbitrary number of times and tests if the computer program samples the "true" extreme point. The computer program evaluates this process by combining these two sampling routines in a Monte Carlo fashion (call this the brute force approach). Of course, without knowing the "true" extreme point *a priori* there is no way of knowing the "true" extreme point was sampled in practice. In this research the "true" solution is known at the outset because the computer program defines the underlying truth model.

5.4 Visits to true Extreme Points via Design

As an alternative to the brute force approach of Monte Carlo sampling this research investigated sampling using a design. This is investigated by applying a Box-Behnken design (3:519) with only one sample at the zero level. The goal is to sample the true extreme point, a design is used to try to minimize the number of samples of the objective function of the linear program. Modifications to the Box-Behnken design were investigated.

5.5 Screening Extreme Points

Evaluating the estimated objective function using either Monte Carlo sampling or a design, requires solving a linear program for every sample of the objective function. Solving multiple linear programs requires a lot of computer time, and for a large problem threatens to make this research impractical. If a way could be found to test each sampled objective function to see if its optimum basis has previously been sampled then a screening method could be applied to reduce the number of linear program solutions required. In this research, only the objective function is stochastic and therefore only optimality, and not feasibility, is an issue. The optimality condition, for a maximization problem, in the general case is:

$$\mathbf{c} - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A} \leq 0 \tag{7}$$

where

$\mathbf{c}$ = objective function

$\mathbf{c}_B$ = coefficient of the basic variables

$\mathbf{B}^{-1}$ = inverse of columns under basic variables

$\mathbf{A}$ = constraint matrix

The specific equation is:

$$\tilde{\mathbf{c}} - \tilde{\mathbf{c}}_B \mathbf{B}^{-1} \mathbf{A} \leq 0 \tag{8}$$

where

$\tilde{\mathbf{c}}$ = the coefficients of the sampled objective function

$\tilde{\mathbf{c}}_B$ = the coefficients of the sampled objective function associated with the basic variables.

As new extreme points are sampled their corresponding $\mathbf{B}^{-1}\mathbf{A}$ matrix and a vector identifying the basic variables are stored in a set, this set is then used to screen new objective function samples. For every new sample of the objective function the computer program cycles through the $\mathbf{B}^{-1}\mathbf{A}$ matrices using Equation 8 until the optimality condition is satisfied, if the optimality condition is not satisfied it solves the linear program to identify a new basis. Using this technique a linear program is solved only once for each unique extreme point sampled.

6 Phase II: Investigation Results

6.1 Introduction

Sampling the estimated objective function to include the "true" optimal extreme point in the sampled set proved an effective method. While it is not possible to identify the "true" optimal extreme point when sampled, it is possible to sample a set of extreme points which include the "true" optimal extreme point.

6.2 "True" Extreme Point Visits w/ Monte Carlo Sampling

Monte Carlo sampling proved a effective method of sampling over a broad range of problems. As is expected, the probability of sampling the "true" optimal extreme point increases as the number of Monte Carlo samples increases. The numbers of Monte Carlo samples required for a given confidence level presumably increases as the number of decisions variables increases and as the error in the estimate increases. This research has not investigated any method to determine *a priori* the number of samples required to achieve a given confidence level. The following tables are typical results for problems involving 4 decision variables. The percent "true" optimal extreme point miss is evaluated by replicating the process 1000 times.

Standard Error	.25	1.25	2.25	3.25	4.25
100 Samples					
% Miss "true"	.8	1.8	2.6	5.2	10.0
200 Samples					
% Miss "true"	1.0	1.6	1.9	2.4	4.25
300 Samples					
% Miss "true"	.8	1.3	1.5	2.3	3.7
500 Samples					
% Miss "true"	.5	.9	1.3	1.9	3.1

Table 1. Monte Carlo Samples

An advantage to Monte Carlo sampling is that if the analyst has the time and resources, and wants to be conservative, the number of Monte Carlo samples could be increased. While Monte Carlo sampling is the least efficient, if the analyst is willing to take enough samples, it could be the most effective in sampling the "true" optimal extreme point.

6.3 "True" Extreme Point Visits w/ Design Sampling

First, a modified Box-Behnken design was used to sample the objective function of the linear program. The modification involved only one sample at the zero level. The sampling was done by varying the estimated objective function coefficients by a percent of the estimated standard deviation (called standard deviation multiplier) in a method prescribed by the design. This approach was possible, in this case, because an orthogonal design was used to sample the original "Black Box Simulation" to estimate the response surface, as a result there are no off-diagonal elements in the variance-covariance matrix. A more complicated method is need if off-diagonal elements were present, but it seems an initial orthogonal design is a reasonable approach. An example of a four-variable Box-Behnken design is found in Appendix B.

Table 2 illustrates the results from a single Box-Behnken design with shown standard deviation multipliers.

Standard Error	.25	1.25	2.25	3.25	4.25
Standard Dev	**1.5**				
% Miss "true"	2.0	3.2	6.7	18.2	30.7
Standard Dev	**2.0**				
% Miss "true"	.9	1.3	7.0	22.3	37.1
Standard Dev	**2.5**				
% Miss "true"	.2	.8	11.5	32.7	41.6
Standard Dev	**3.0**				
% Miss "true"	.1	1.5	18.4	38.9	50.6

Table 2. Single Box-Behnken Design

Tests using this single Box-Behnken design showed limited success. It appears this single design is inadequate to sample the "true" extreme point.

The second modification to the standard Box-Behnken design was to double the length of the design by sampling at each design point twice. For every identical pair of design points different standard deviation multipliers were used. In effect, a three-level design was transformed into a pseudo five-level design. It is not a true five-level design because each design point has only three levels, it is really the same design run twice with two different standard deviation multipliers. Table 3 shows results of this technique.

Standard Error	.25	1.25	2.25	3.25	4.25
Standard Dev's	**1, 1.5**				
% Miss "true"	4.0	5.5	6.5	7.5	20.5
Standard Dev's	**1.5, 2**				
% Miss "true"	.9	1.5	3.6	9.9	20.2
Standard Dev's	**1.5, 2.5**				
% Miss "true"	.2	.1	2.8	10.3	19.6
Standard Dev's	**1.5, 3**				
% Miss "true"	.25	0.0	1.65	9.1	20.3
Standard Dev's	**1, 2.5**				
% Miss "true"	.2	.1	2.4	9.9	17.1
Standard Dev's	**1, 2**				
% Miss "true"	.9	1.5	3.6	8.1	16.0
Standard Dev's	**1, 3**				
% Miss "true"	0.0	0.0	2.2	11.0	20.3

Table 3. Double Box-Behnken Type Design

The double Box-Behnken design (sampling the objective function 49 times) shows promise. Results with the double Box-Behnken design are superior to sampling in a Monte Carlo fashion 49 times.

Standard Error	.25	1.25	2.25	3.25	4.25
% Miss "true"	1.2	4.0	5.8	12.3	18.6

Table 4. 49 Monte Carlo Samples

Results over a broad range of problems indicate this design is superior, but not dramatically, to an equivalent number of Monte Carlo samples. In general, either case fails to give confidence in the results.

The next modification includes adding a third Box-Behnken design to the previous two designs and sampling it at a different standard deviation, this is a pseudo seven-level design. In essence, this is equivalent to sampling from three consecutive designs.

Standard Error	.25	1.25	2.25	3.25	4.25
Standard Dev's	**.5, 1.5,**	**2.5**			
% Miss "true"	0.3	0.2	1.4	5.8	10.6
Standard Dev's	**1.0, 2.0,**	**3.0**			
% Miss "true"	0.1	0.2	0.8	4.8	11.5
Standard Dev's	**1.0, 1.75**	**2.5**			
% Miss "true"	0.3	0.2	1.1	4.4	9.9
Standard Dev's	**.5, 1.75,**	**3.0**			
% Miss "true"	0.1	0.2	0.8	5.0	10.6
Standard Dev's	**1.0, 1.75**	**2.5**			
% Miss "true"	0.9	3.2	7.4	15.5	26.1

Table 5. Triple Box-Behnken Type Design

The triple Box-Behnken design had good results, but required more samples. In this case, the triple Box-Behnken design (with four decision variables) was sampled 73 times. As a comparison, the results of 73 Monte Carlo samples are presented in Table 6.

Standard Error	.25	1.25	2.25	3.25	4.25
% Miss "true"	1.0	2.2	4.3	7.7	12.5

Table 6. 73 Monte Carlo Samples

Again there is an advantage to the design over the equivalent number of Monte Carlo samples. The main advantage to a Monte Carlo approach is that the number of samples can be arbitrarily increased to achieve the confidence desired, this may be desirable if a higher confidence in the solution in needed than is possible with this design. To this point, each design was an improvement over an equivalent number of Monte Carlo samples, but no design gave a high success rate at higher noise levels.

In an effort to improve the success rates with higher levels of noise another type of modification to the basic Box-Behnken design was investigated. In this case, the basic structure at each design point was modified. Instead of sampling at the design points using a three-level approach of 1, -1, or 0, this new design was a true five-level design where each design point was sampled with some combination of 1, -1, .5, -.5, or 0. This modification doubles the length of the design and at each design point alternatively samples form either 1 or .5. An example of this new design is found in Appendix B.

Standard Error	.25	1.25	2.25	3.25	4.25
Standard Dev	**1.5**				
% Miss "true"	5.1	6.8	10.3	12.5	15.5
Standard Dev	**2.0**				
% Miss "true"	1.7	3.1	4.0	6.3	10.8
Standard Dev	**2.5**				
% Miss "true"	.5	1.8	2.1	5.3	11.2
Standard Dev	**3.0**				
% Miss "true"	0.0	.6	1.7	5.7	12.3

Table 7. Single 5-level Box-Behnken Type Design

The single modified 5-level Box-Behnken design has 49 design point, the same number as the double Box-Behnken design presented in Table 3. The 5-level design has a higher success rate in sampling the "true" optimal extreme point than either the double Box-Behnken design, or a equivalent number of Monte Carlo samples. The 5-level Box-Behnken design represents an improvement when sampling at higher noise levels, but the errors could still be considered significant.

A further modification attempts to decrease the errors in sampling the "true" optimal extreme point by doubling the design and choosing a different standard deviation multiplier for the second half of the design. This modification is analogous to the change creating the double Box-Behnken design. This design creates a pseudo nine-level design. The results of 1000 replications of this design are contained in Table 8.

Standard Error	.25	1.25	2.25	3.25	4.25
Standard Dev	**.5, 1.75**				
% Miss "true"	2.67	4.0	6.6	9.2	12.9
Standard Dev	**1.0, 2.0**				
% Miss "true"	1.8	2.9	4.5	6.2	10.2
Standard Dev	**1.5, 2.5**				
% Miss "true"	0.8	1.2	1.9	3.2	6.1
Standard Dev	**1.5, 3.0**				
% Miss "true"	0.2	0.3	1.0	2.6	6.8

Table 8. Double 5-level Box-Behnken Type Design

The double modified 5-level Box-Behnken design gave excellent results. This design gave the best results for methods with about 97 samples, and it is competitive with a Monte Carlo method of 200 samples.

In the next modification another modified 5-level design is added and sampled at a different standard deviation. This pseudo 13-level design (four variables) has 145 design points. The results are found in Table 9 and show excellent results.

Standard Error	.25	1.25	2.25	3.25	4.25
Standard Dev	**1.0, 2.0,**	**3.0**			
% Miss "true"	0.1	0.3	0.3	1.9	3.0
Standard Dev	**1.5, 2.5,**	**3.5**			
% Miss "true"	0.0	0.1	0.1	1.1	2.9
Standard Dev	**1.5, 2.75**	**4.0**			
% Miss "true"	0.0	0.0	0.1	0.5	1.9
Standard Dev	**1.5, 3.0,**	**4.5**			
% Miss "true"	0.0	0.0	0.0	0.5	2.2

Table 9. Triple 5-level Box-Behnken Type Design

145 samples of the triple 5-level Box-Behnken design was superior to all other designs and even superior to 500 Monte Carlo samples. This design provides excellent sampling in an relatively efficient manner. The main drawback is that it requires 145 samples with only four variables. If this full design is run its length will double with every added variable. As a possible way to offset the time required to solve this many linear programs this research investigated a method of screening sampled objective functions to decrease the computations required.

Another interesting consideration is the number of extreme points visited with different sampling techniques. If one sampling method provided high accuracy, but required may more extreme points to be sampled, then it might not be the best design to employ. Fortunately, no design greatly increased the number of extreme points sampled. Table 10 illustrates the total unique extreme points sampled for 200 Monte Carlo samples and two design, these results are typical of all sampling options.

Standard Error	.25	1.25	2.25	3.25	4.25
200 Monte Carlo					
# unique ext. points	3	4	6	8	9
Double 5-level Box-Behnken Type	**1.5, 2.5**				
# unique ext. points	2	3	5	8	8
Triple 5-level Box-Behnken Type	**1.5, 2.75,**	**4.0**			
# unique ext. points	2	4	5	7	8

Table 10. Total Unique Extreme Points Sampled by Case

6.4 Screening Extreme Points

The main drawback to using the above approaches is the need to solve a linear program for every sampled objective function. Screening the new objective function samples proved an efficient technique. With this technique, a linear program is solved only once for each sampled extreme point. The improved efficiency will vary from problem to problem and will also depend on the number of objective function samples, but improvement can be measured in orders of magnitude. Applying this technique greatly increases the practicality and efficiency of the research. Using this screening procedure makes a strong case for using the triple 5-level Box-Behnken design approach with a large number of samples.

In this research the IMSL Fortran Library was used to evaluate the revised simplex method, but it does not return a $\mathbf{B}^{-1}$ matrix, only values for the optimal value (Z^*), the primal and dual solutions of the decision variables. To employ this screening technique the $\mathbf{B}^{-1}$ matrix must be found. The **B** matrix was found by choosing the columns under the basic variables. The values of slack variables are not given; therefore, when slack variables are basic the computer program uses the principle of complementary slackness to identify the values of the corresponding columns in the **B** matrix.

7 Phase III: Selecting the Optimal Extreme Point

7.1 Introduction

Phase II established a method to sample extreme points with the goal of including the "true" extreme point in the sample. Phase III's aim is to pick the "true" optimal extreme point from the population of sampled extreme points. To do this, consider the extreme points as settings of the input decision variables to the simulation.

7.2 Starting Hypothesis

After identifying the feasible extreme points, the linear program is no longer needed, and the decision variable settings at any extreme point are used as input to the simulation to estimate Z^*. Once a set of decision variable settings is selected the problem becomes selecting the "best" option. This research starts with the premise that given a set of sampled extreme points containing the "true" optimal extreme point, it can be identified as the optimal given enough sampling:

$$\Phi_i = E(EP_i) \text{ as } N \to \infty \tag{9}$$

If $EP^* \subset EP$ then

$$E(EP^*) = \max_i \Phi_i = Z^* \tag{10}$$

where

N = number of samples from simulation (i.e., simulation runs)

Φ_i = Value of "Truth Model" at ith extreme point

EP^* = "true" optimal extreme point

EP = set of sampled extreme points

EP_i = ith extreme point in sample (arbitrary ordering)

Using this method it may be possible to both identify the "true" optimal settings for the decision variables (extreme point) and an unbiased estimate for the optimal solution Z^*.

But, this method requires many samples from the simulation and only considers the means and not the distributions around the means.

7.3 Ranking and Selection of Decision Variable Sets

Law and Kelton present ranking and selection procedures that offers an alternative to the brute force method presented above (8:596). The analyst may be interested in three selection approaches. First, the procedure is based on selecting the best of the k decision variable settings. Second, as an initial screening procedure, selecting a subset of size m that contains the best of the k decision variable settings. Third, selecting the best m alternatives from the k decision variable settings, this approach would offer greater flexibility for the decision maker by providing more options.

7.3.1 Selecting the Best of k Systems

Previously, $\hat{Z}^*$ represented the estimated optimal solution from the *linear program.* Now, let the Z_{ij}'s represent point estimates from the *simulation* for the jth replication of the ith decision variable set and $\mu_i = E(Z_{ij})$. This approach assumes the Z_{ij}'s are independent (8:596). Law and Kelton define a method of finding the smallest expected response; this research focuses on the largest expected response defining μ_{i_ℓ} to be the ℓth largest of the μ_i's and

$$\mu_{i_1} \geq \mu_{i_2} \geq \cdots \geq \mu_{i_k} \tag{11}$$

we want $P(CS) \geq P^*$ (12)

provided $\mu_{i_1} - \mu_{i_2} \geq d^*$ (13)

and the minimal CS probability is $P^* > 1/k$ (14)

where

CS = correct solution

$d^* > 0$ defines the "indifference" specified by the analyst.

Law and Kelton present a procedure originally developed by Dudewicz and Dalal (1975)

> [Involving] "two-stage" sampling from each of the k systems. In the first stage we make a fixed number of replications of each system, then use the resulting variance estimates to determine how many more replications from each system are necessary in a second stage of sampling in order to reach a decision. It must be assumed that the [Z_{ij}'s] are normally distributed, but (importantly) we need *not* assume that the values of $\sigma_i^2 = \mathrm{Var}(Z_{ij})$ are known; nor do we have to assume that the σ_i^2 are the same for different i's. (Assuming known or equal variances is very unrealistic when simulating real systems.) The procedure's performance should be robust to departures from the normality assumption, especially if the [Z_{ij}'s] are averages. (8:596)

Law and Kelton define the first-stage sampling with $n_0 \geq 2$ replications of each k decision variable sets and define:

$$\bar{Z}_i^{(1)}(n_0) = \frac{\sum_{j=1}^{n_0} Z_{ij}}{n_0} \qquad (15)$$

$$S_i^2(n_0) = \frac{\sum_{j=1}^{n_0} [Z_{ij} - \bar{Z}_i^{(1)}(n_0)]^2}{n_0 - 1} \qquad (16)$$

for i = 1, 2, . . . , k. N is total sample size needed for system i

$$N_i = \max\left\{ n_0 + 1, \left\lceil \frac{h_1^2 S_i^2(n_0)}{(d^*)^2} \right\rceil \right\} \qquad (17)$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to the real number x, and h_1 (which depends on k, P^*, and n_0) is a constant that can be obtained from Table 1

(Appendix C). Next, make $N_i - n_0$ more replications of system i ($i = 1,2,\ldots,k$) and solve for the second-stage sample means by

$$\bar{Z}_i^{(2)}(N_i - n_0) = \frac{\sum_{j=n_0+1}^{N_i} Z_{ij}}{N_i - n_0} \tag{18}$$

Define the weights as

$$W_{i1} = \frac{n_0}{N_i}\left[1 + \sqrt{1 - \frac{N_i}{n_0}\left(1 - \frac{(N_i - n_0)\,(d^*)^2}{h_1^2\, S_i^2(n_0)}\right)}\right] \tag{19}$$

and $W_{i2} = 1 - W_{i1}$, for $i = 1,2,\ldots,k$. Also, define the weighted sample means as

$$\tilde{Z}_i(N_i) = W_{i1}\,\bar{Z}_i^{(1)}(n_0) + W_{i2}\,\bar{Z}_i^{(2)}(N_i - n_0) \tag{20}$$

and select the decision variable setting (extreme point) with the largest $\tilde{Z}_i(N_i)$ (8:597). Law and Kelton conclude:

> The choice of P^* and d^* depend on the analyst's goals and the particular system under study; specifying them might be tempered by the computing cost of obtaining a large N_i associated with a large P^* or small d^*. However, choosing n_0 is more troublesome, and we can only say, on the basis of our experiments and various statements in the literature, that n_0 be at least 20. (8: 597-598)

7.3.2 Selecting a Subset m Containing the Best of k Decision Variables

There may be cases where EP (the set of sampled extreme points)1 is large and the above approach would require too much computer time and effort; here an initial screening procedure could be useful.

Now consider selecting a subset of size m containing the best from k decision variable settings (i.e., EP_i where $i = 1, 2, \ldots, k$) where m is specified by the analyst. This is that same situation as above except "correct selection (CS) is defined to mean that the subset of size m that is selected contains a system [extreme point] with mean μ_{i_1} and we want $P(CS) \geq P^*$ provided that $\mu_{i_1} - \mu_{i_2} \geq d^*$; here we must have $1 \leq m \leq k-1$, $P^* > m/k$, and $d^*>0$ (8:599)." Also, replace h1 by h_2 that depends not only on k, P^*, and n_0, but also on m (see Table 2 Appendix C).

> Then we make $N_i - n_0$ more replications, from the second stage means $\bar{Z}_i^{(2)}(N_i - n_0)$, weights Wi1 and Wi2, and weighted sample means $\tilde{Z}_i(N_i)$, [exactly as before]. Finally, we define the selected subset to consist of the m systems corresponding to the m smallest values of the $\tilde{Z}_i(N_i)$'s. (8:599)

As m increases "considerably fewer replications" are required than when m = 1 (8:600).

7.3.2 Selecting the Best m of k Decision Variable Settings

Using the m best of k decision variable settings approach, the analyst can provide the best m alternatives to the decision maker giving him a broader base for a decision. This approach is very similar to the above two approaches except the subset of size m equals the largest expected responses $\mu_{i_1}, \mu_{i_2}, \ldots, m_{i_m}$ (this set is not ranked ordered). Here $P(CS) \geq P^*$ provided $\mu_{i_m} - \mu_{i_{m+1}} \geq d^*$. Also, $P^* > m!(k-m)!/k!$ and replace h_2 by h_3 (see Table 3 Appendix C).

7.4 Histogram Comparison

The ranking and selection procedure above presents a method of evaluating the expected value of different decision variable settings, but choosing the "best" solution often involves more than just identifying the largest expected value.

Histograms are plotted using all simulation samples from the best m alternatives. The histogram can the aid the decision maker by visually representing the possible realizations of the process at given settings. Two important advantages are: avoiding risk by choosing the smallest variance, and illustrating nearly equivalent alternatives and allowing the decision maker to consider factors not captured by the model. For instance, figure 11 illustrates possible histograms from the top three decision variable sets.

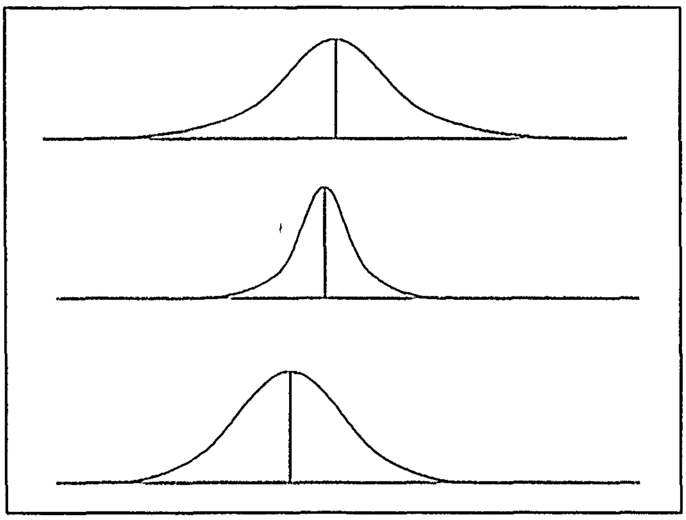


Figure 11 Histogram Comparison

It is not clear which is the best alternative. The top plot has the highest mean, but a risk averse decision maker may choose the second option to avoid the possible down side of the first option. In either case, a visual representation presents the decision maker with a broader knowledge base from which to make a decision.

8 Phase III Investigation Results

A ranking and selection procedure was used to analyze the extreme points sampled by the double Box-Behnken type design when the standard error equals 3.25. The sampled extreme points are in Table 11, and include the actual Z^* at each extreme point which, of course, would not be known in practice.

Extreme Point	x1	x2	x3	x4	True Z^*
1	4	0	0.666667	6.6666667	215.333
2	1	11	0	0	212
3	2	10	0	0	210
4	2	8	0	2	216
5	8	0	2	0	166
6	0	8	2	0	182
7	0	0	4.666667	2.666667	147.333
8	0	0	6	0	118

Table 11. Unique Extreme Points Sampled

Because there are only eight extreme points no screening is needed, and the ranking and selection of the best m out of k alternatives was used. Lets assume the best three alternatives are desired at a 90% confidence level (i.e., $h_3 = 3.532$) with d= 4. Where d is the minimum separation between extreme point sample means desired, and like the confidence level is chosen by the analyst. Knowing the true separation between Z^* it is clear d is too high, here it is chosen a little high to illustrate the robustness of the process. There will always be a trade off between the confidence level desired, the value of d, and the number of samples (N_i) required at each extreme point. As the confidence level increases, or the value of d decreases, the number of samples required increases. Table 12 correctly identifies the top three alternatives, recall that the order of the three alternatives is not guaranteed. In this case, the decision maker would choose between extreme points one, two and four.

i	$\bar{Z}_i^1(20)$	$S_i^2(20)$	N_i	$\bar{Z}_i^2(N_i\text{-}20)$	W_{i1}	W_{i2}	$\tilde{Z}_i(N_i)$
1	217.876	154.707	121	217.135	.186	.814	217.273
2	206.517	118.576	93	212.157	.247	.753	210.767
3	209.244	122.273	96	208.107	.242	.758	208.837
4	214.886	254.121	199	215.379	.12	.88	215.32
5	167.00	182.121	142	164.912	.142	.858	165.209
6	186.924	112.299	88	179.201	.257	.743	181.186
7	145.86	165.889	130	146.701	.18	.82	146.55
8	113.123	135.033	106	116.445	.221	.779	115.791

Table 12. Selecting the Three (w/ d=4) Best of the Eight Extreme Points

Table 13 shows a case where a different approach is taken. The initial estimates of the means $\bar{Z}_i^1(20)$ show there appears to be a natural division between the first four extreme points and the last four. Because of this natural division it might be advantageous to choose the best four extreme points out of the set of eight. In this case, at a 90% confidence level h_3 equals 3.571 and d is again chosen equal to four.

i	$\bar{Z}_i^1(20)$	$S_i^2(20)$	N_i	$\bar{Z}_i^2(N_i\text{-}20)$	W_{i1}	W_{i2}	$\tilde{Z}_i(N_i)$
1	217.876	154.707	124	216.534	.189	.811	216.788
2	206.517	118.576	95	212.193	.24	.76	210.831
3	209.244	122.273	98	208.554	.234	.766	208.716
4	214.886	254.121	203	215.129	.133	.887	215.102
5	167.00	182.121	146	164.80	.163	.837	165.159
6	186.924	112.299	90	179.182	.253	.747	181.143
7	145.86	165.889	134	146.854	.178	.822	146.677
8	113.123	135.033	108	116.624	.208	.792	115.895

Table 13. Selecting the Four (w/ d=4) Best of the Eight Extreme Points

Table 13 correctly identifies to top four alternatives. Even overestimating d, this method proved effective. Table 14 illustrates another approach, the $\bar{Z}_i^1(20)$ seem to show two distinct groups. This can be exploited, instead of d=4 let us choose d=6, this is done with the goal of separating the groups and then getting a feel for the rankings. This approach might be taken if multiple replications are difficult to make (note the decrease in N_i from

Table 13 to Table 14). Also, this approach illustrates another advantage of this technique, using the ranking and selection procedure it is possible to identify the top m competing alternatives. In other words, this helps identify the number of roughly equivalent alternatives (i.e., the value for m).

i	$\bar{Z}_i^1(20)$	$S_i^2(20)$	N_i	$\bar{Z}_i^2(N_i\text{-}20)$	W_{i1}	W_{i2}	$\tilde{Z}_i(N_i)$
1	217.876	154.707	55	215.661	.393	.607	216.531
2	206.517	118.576	43	213.92	.542	.458	209.908
3	209.244	122.273	44	208.454	.517	.483	208.863
4	214.886	254.121	91	215.833	.263	.737	215.583
5	167.00	182.121	65	162.854	.348	.652	164.296
6	186.924	112.299	40	179.464	.537	.463	183.472
7	145.86	165.889	59	148.435	.369	.631	147.484
8	113.123	135.033	48	115.913	.446	.554	114.669

Table 14. Selecting the Four (w/ d=6) Best of the Eight Extreme Points

After choosing a ranking and selection method, the confidence level, and d then this research recommends using the data obtained through the procedure to create a histogram. The histogram is a way to aid the decision maker. In this example, all the actual variances are equal, but this method has its strengths when the variances are different. Figure 12 illustrates the histograms of the top four alternatives, the dotted vertical lines represent $\tilde{Z}_i(N_i)$ for each alternative. An alternative to presenting a histogram of the data is the plot the nomal probability curve defined by the estimated mean and variance.

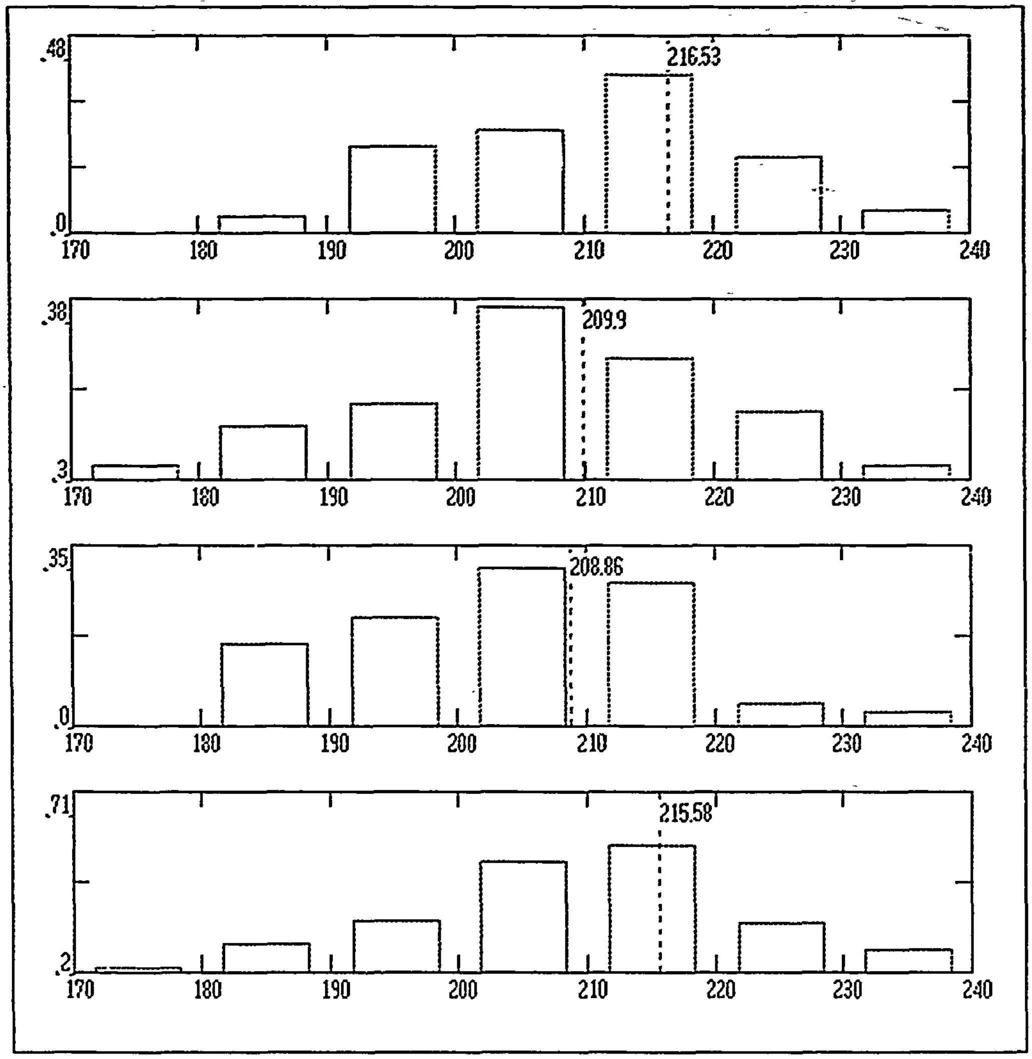


Figure 12 Sample Case Histogram Comparison

At this point the choice is up to the decision maker.

9 Conclusions and Recommendations

9.1 Introduction

This research presents a superior method to the traditional approach of estimating a response surface and then using it as the objective function of a linear program. Over multiple realizations the traditional approach will overestimate the true mean response, and it is unlikely that the "true" optimal extreme point will be chosen. Small variance in the estimates of the response surface coefficients can lead to large variance in the estimation of Z^* and a low probability of choosing the correct optimal extreme point EP^*. By using the screening procedure this general procedure may become practical for general application.

9.2 Variance Reduction

The results of this research clearly lead to the conclusion that some kind of variance reduction techniques applied to the simulation would greatly benefit the analyst. If the analyst chooses to use the traditional method of solving this kind of problem (with only one realization of the process) variance reduction procedures appear to be critical if he hopes to have any confidence in the solution. If the analyst chooses to follow the approach recommended in this research variance reduction will play a key role in minimizing the number of extreme points sampled and aiding in the comparison between competing extreme points.

Please refer to Law and Kelton (1991) for explanation of how to apply variance reduction techniques. Some techniques that may be appropriate here are: multiple replications, common random numbers, antithetic random numbers, and control variates.

9.3 Three Step Process

As stated earlier, the purpose of this research was to investigate the traditional approach of solving constrained optimization problems through simulation and linear programming. Problems have been identified and a possible solution offered. Figure 13 illustrates a standard estimation of a response surface and its variance-covariance matrix.

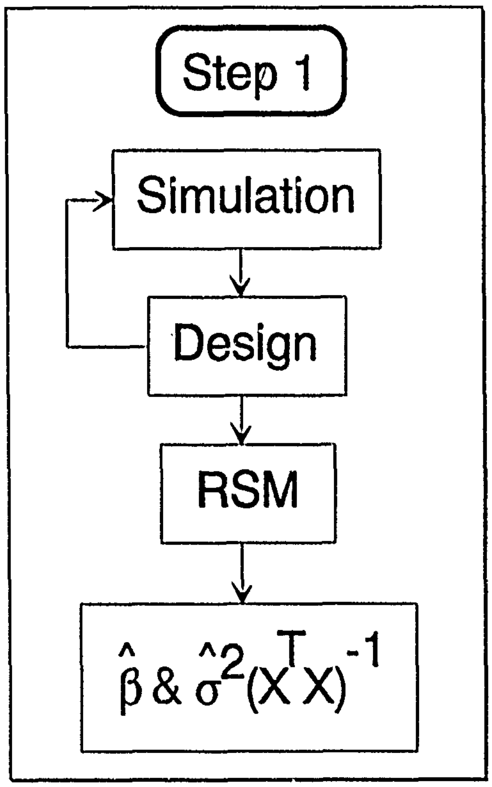


Figure 13 Step 1

Figure 14 illustrates a Monte Carlo or design procedure (using screening) to sample the objective function with the goal of including the true optimal extreme point (EP^*) in the sampled set. It appears the superior choice is to uses the screening procedure and then sample from the objective function using a triple 5-level Box-Behnken type design.

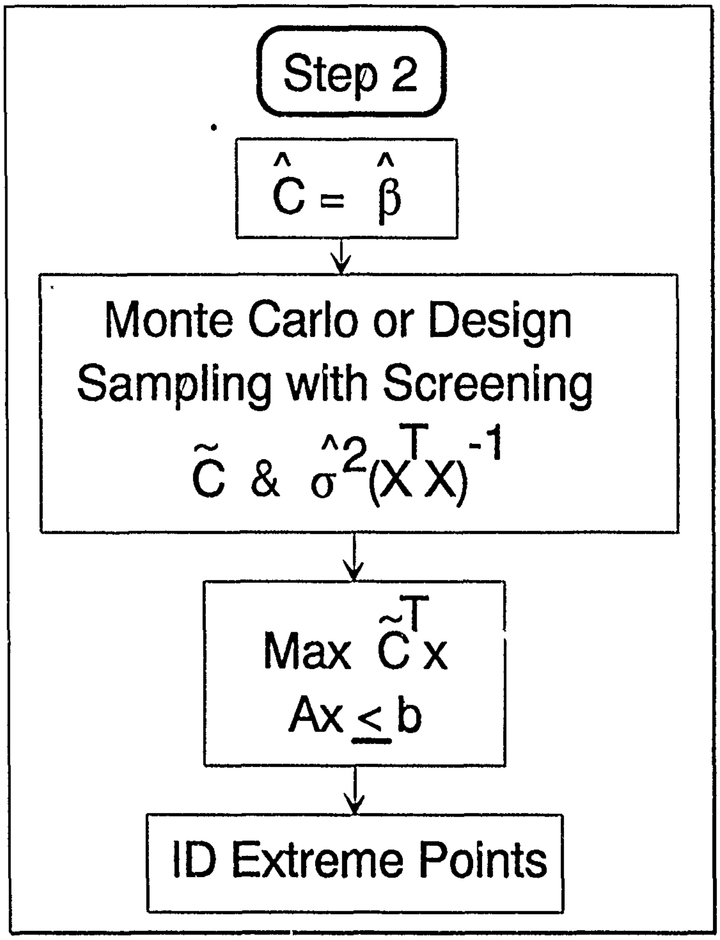


Figure 14 Step 2

Figure 15 illustrates how to identify the "true" optimal extreme point EP^* (with a given probability) and present the information to a decision maker in both numeric and visual form.

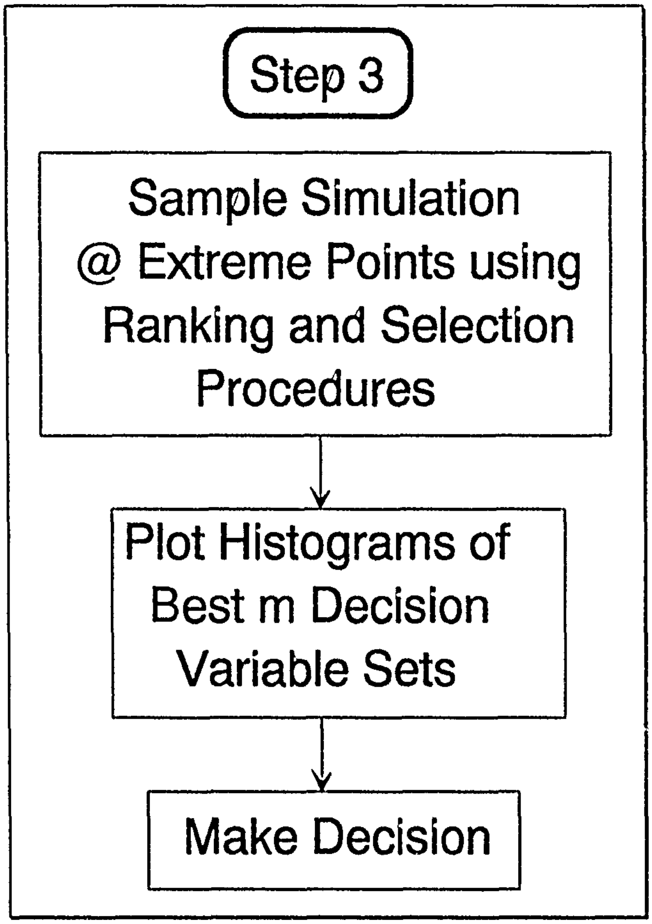


Figure 15 Step 3

9.4 Further Research Areas

This research could be continued in the following areas:

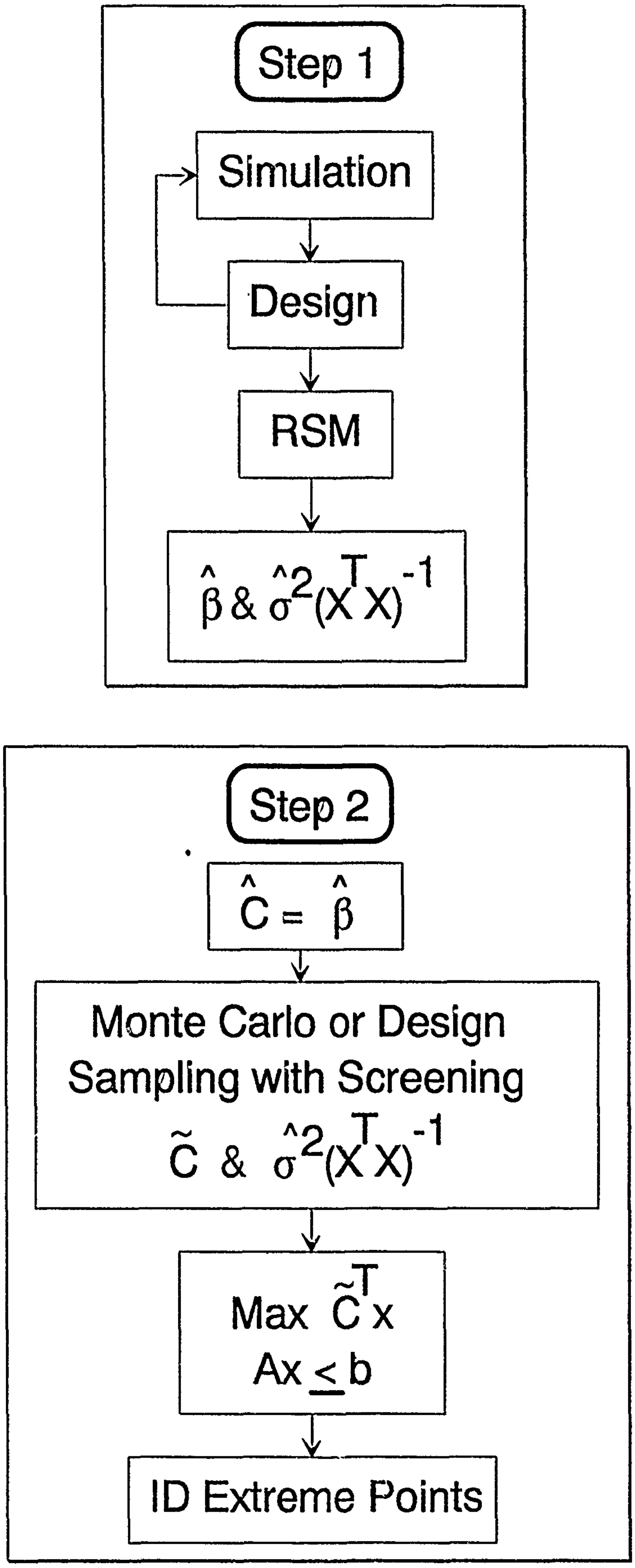
1. Investigate alternative sampling designs with the goal of decreasing the number of design points and increasing the "true" optimal extreme point sampling accuracy.

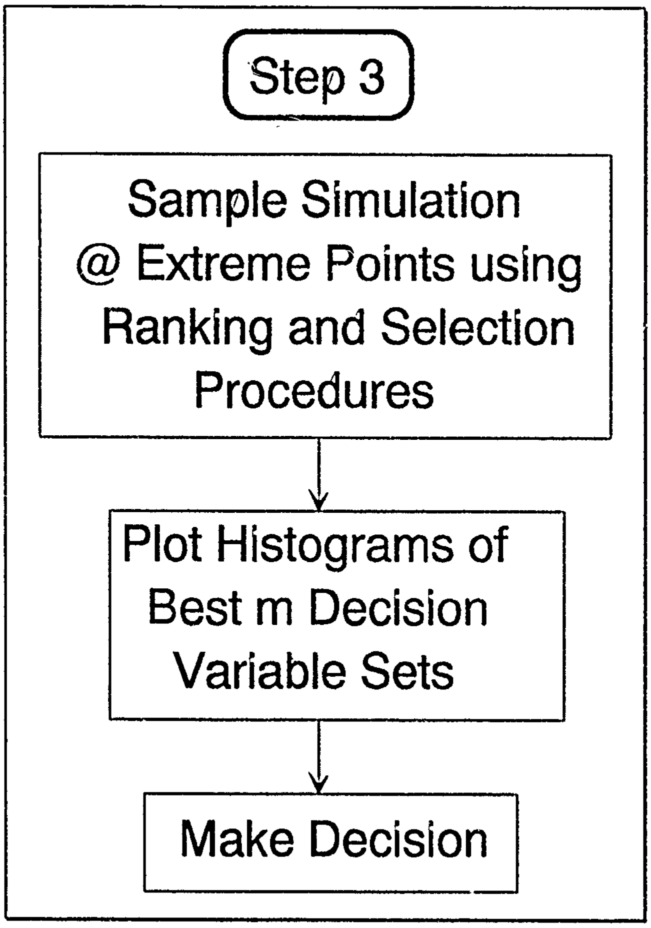
2. Investigate an alternative structure to decrease the number of design points in the triple modified 5-level Box-Behnken design.

3. Investigate a method to determine *a priori* the number of samples that must be taken to achieve a given confidence level.

4. Investigate an alternative to Law and Kelton's ranking and selection procedure presented in this research.

Appendix A: Illustration of Three Step Process


Step 1
Simulation
Design
RSM
$\hat{\beta}$ & $\hat{\sigma}^2(X^TX)^{-1}$
Step 2
$\hat{C} = \hat{\beta}$
Monte Carlo or Design
Sampling with Screening
$\tilde{C}$ & $\hat{\sigma}^2(X^TX)^{-1}$
Max $\tilde{C}^T x$
$Ax \leq b$
ID Extreme Points


Step 3
Sample Simulation @ Extreme Points using Ranking and Selection Procedures
Plot Histograms of Best m Decision Variable Sets
Make Decision

Appendix C: Box Behnken Type Designs

-1	-1	0	0
1	-1	0	0
-1	1	0	0
1	1	0	0
0	0	-1	-1
0	0	1	-1
0	0	-1	1
0	0	1	1
-1	0	0	-1
1	0	0	-1
-1	0	0	1
1	0	0	1
0	-1	-1	0
0	1	-1	0
0	-1	1	0
0	1	1	0
-1	0	-1	0
1	0	-1	0
-1	0	1	0
1	0	1	0
0	-1	0	-1
0	1	0	-1
0	-1	0	1
0	1	0	1
0	0	0	0

Single Box-Behnken Design (four variables)

-1.0	-0.5	0.0	0.0
1.0	-0.5	0.0	0.0
-1.0	0.5	0.0	0.0
1.0	0.5	0.0	0.0
0.0	0.0	-1.0	-0.5
0.0	0.0	1.0	-0.5
0.0	0.0	-1.0	0.5
0.0	0.0	1.0	0.5
-1.0	0.0	0.0	-0.5
1.0	0.0	0.0	-0.5
-1.0	0.0	0.0	0.5
1.0	0.0	0.0	0.5
0.0	-1.0	-0.5	0.0
0.0	1.0	-0.5	0.0
0.0	-1.0	0.5	0.0
0.0	1.0	0.5	0.0
-1.0	0.0	-0.5	0.0
1.0	0.0	-0.5	0.0
-1.0	0.0	0.5	0.0
1.0	0.0	0.5	0.0
0.0	-1.0	0.0	-0.5
0.0	1.0	0.0	-0.5
0.0	-1.0	0.0	0.5
0.0	1.0	0.0	0.5
-0.5	-1.0	0.0	0.0
0.5	-1.0	0.0	0.0
-0.5	1.0	0.0	0.0
0.5	1.0	0.0	0.0
0.0	0.0	-0.5	-1.0
0.0	0.0	0.5	-1.0
0.0	0.0	-0.5	1.0
0.0	0.0	0.5	1.0
-0.5	0.0	0.0	-1.0
0.5	0.0	0.0	-1.0
-0.5	0.0	0.0	1.0
0.5	0.0	0.0	1.0
0.0	-0.5	-1.0	0.0
0.0	0.5	-1.0	0.0
0.0	-0.5	1.0	0.0
0.0	0.5	1.0	0.0
-0.5	0.0	-1.0	0.0
0.5	0.0	-1.0	0.0
-0.5	0.0	1.0	0.0
0.5	0.0	1.0	0.0
0.0	-0.5	0.0	-1.0
0.0	0.5	0.0	-1.0
0.0	-0.5	0.0	1.0
0.0	0.5	0.0	1.0
0.0	0.0	0.0	0.0

Box-Behnken Type Design with 5 levels (four variables)

Appendix C: Constants for the Selection Procedures

P*	n_0	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9	k = 10
0.90	20	1.896	2.342	2.583	2.747	2.870	2.969	3.051	3.121	3.182
0.90	40	1.852	2.283	2.514	2.669	2.758	2.878	2.954	3.019	3.076
0.95	20	2.453	2.872	3.101	3.258	3.377	3.472	3.551	3.619	3.679
0.95	40	2.386	2.786	3.003	3.150	3.260	3.349	3.422	3.484	3.539

(8:606)

Table 15. Value for h_1

(for m =1, use Table 1)

m	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9	k = 10
			P^*=.90	n_0=20				
2	1.137	1.601	1.860	2.039	2.174	2.282	2.373	2.450
3		0.782	1.243	1.507	1.690	1.830	1.943	2.038
4			0.556	1.012	1.276	1.461	1.603	1.718
5				0.392	0.843	1.105	1.291	1.434
6					0.265	0.711	0.971	1.156
7						0.162	0.603	0.861
8							0.075	0.512
9								N/A
			P^*=.90	n_0=40				
2	1.114	1.570	1.825	1.999	2.131	2.237	2.324	2.399
3		0.763	1.219	1.479	1.660	1.798	1.909	2.002
4			0.541	0.991	1.251	1.434	1.575	1.688
5				0.381	0.824	1.083	1.266	1.408
6					0.257	0.693	0.950	1.133
7						0.156	0.587	0.841
8							0.072	0.497
9								N/A
			P^*=.95	n_0=20				
2	1.631	2.071	2.321	2.494	2.625	2.731	2.819	2.894
3		1.256	1.697	1.952	2.131	2.267	2.378	2.470
4			1.021	1.458	1.714	1.894	2.033	2.146
5				0.852	1.284	1.539	1.720	1.860
6					0.721	1.149	1.402	1.583
7						0.615	1.038	1.290
8							0.526	0.945
9								0.449
			P^*=.95	n_0=40				
2	1.591	2.023	2.267	2.435	2.563	2.665	2.750	2.823
3		1.222	1.656	1.907	2.082	2.217	2.325	2.415
4			0.990	1.420	1.672	1.850	1.987	2.098
5				0.824	1.248	1.499	1.678	1.816
6					0.695	1.114	1.363	1.541
7						0.591	1.004	1.252
8							0.505	0.913
9								0.430

(8:606)

Table 16. Value for h_2

(for m =1, use Table 1)

m	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9	k = 10
			P^*=.90	n_0=20				
2	2.342	2.779	3.016	3.177	3.299	3.396	3.477	3.546
3		2.583	3.016	3.251	3.411	3.532	3.629	3.709
4			2.747	3.177	3.411	3.571	3.691	3.787
5				2.870	3.299	3.532	3.691	3.811
6					2.969	3.396	3.629	3.709
7						3.051	3.477	3.709
8							3.121	3.546
9								3.182
			P^*=.90	n_0=40				
2	2.283	2.703	2.928	3.081	3.195	3.285	3.360	3.424
3		2.514	2.928	3.151	3.302	3.415	3.505	3.579
4			2.669	3.081	3.302	3.451	3.564	3.653
5				2.785	3.195	3.415	3.564	3.675
6					2.878	3.285	3.505	3.653
7						2.954	3.360	3.579
8							3.019	3.424
9								3.076
			P^*=.95	n_0=20				
2	2.872	3.282	3.507	3.662	3.779	3.873	3.952	4.019
3		3.101	3.507	3.731	3.885	4.001	4.094	4.172
4			3.258	3.662	3.885	4.037	4.153	4.246
5				3.377	3.779	4.001	4.153	4.269
6					3.472	3.873	4.094	4.246
7						3.551	3.952	4.172
8							3.619	4.019
9								3.679
			P^*=.95	n_0=40				
2	2.786	3.175	3.386	3.530	3.639	3.725	3.797	3.858
3		3.003	3.386	3.595	3.738	3.845	3.931	4.002
4			3.150	3.530	3.738	3.879	3.986	4.071
5				3.260	3.639	3.845	3.986	4.092
6					3.349	3.725	3.931	4.071
7						3.422	3.797	4.002
8							3.484	3.858
9								3.539

(8:606)

Table 17. Value for h_3

Appendix D: Main Computer Program Listing

```
      PROGRAM RSMLP

C  Written by 1LT R. Garrison Harvey  March 1, 1992
C  This program investigates the how the noise in the estimation of
C  a response surface impacts the estimation of the optimal solution
C  when the response surface is used as an objective function of a
linear
C  program.
C
C  This program is a research tool, but with the elimination of
C  unneeded procedures could be used in practice.
C
C  This program has the following main loops:
C
C  Noise loop (NL times)
C     Response surface (obj. function) sampling (DRUNS times)
C         Sampling of the objective function of the LP to
C          identify the "true" optimal extreme point (RUNS times)
C

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCCCCCCCCCC  VARIABLES FOR THE LP PORTION CCCCCCCCCCCCCC

       INTEGER    LDA, M, NVAR, RUNS, BCDIM, EPDIM
       PARAMETER  (M=3,NVAR=4,LDA=M,RUNS=149,BCDIM=20,EPDIM=20)
       INTEGER DP, DRUNS, TRUNS, NVARY,NINT,NL,STEP
       PARAMETER (DP=16, DRUNS=1000, TRUNS=RUNS*DRUNS, NVARY=NVAR+1)
       PARAMETER (NINT=20, NL=5, STEP=50)

C !!!! When changing the parameters remember to change the subroutines

       INTEGER  IRTYPE(M), NOUT
       REAL   A(M,NVAR), B(M), C(NVAR),OBJTRUE(NVAR), DSOL(M)
       REAL   XLB(NVAR), XSOL(NVAR), XUB(NVAR), OBJ,CONSTANT
       REAL   XSOLT(NVAR), OPTT
       EXTERNAL  DLPRS, SSCAL, UMACH

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCC VARIABLES FOR THE MULTIVARIATE NORMAL GENERATION CCC

       INTEGER   IRANK, ISEED
       REAL  R(RUNS,NVAR),RSIG(NVAR,NVAR)
       EXTERNAL CHFAC, RNMVN, RNSET, UVSTA, RNNOA,RNNOF
       EXTERNAL CORVC, WRRRN, RCOV,HHSTP, OWFRQ, PROBP,WROPT
       EXTERNAL SCOLR,LINRG,MRRRR

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCCC  VARIABLES FOR THE MONTE CARLO SIMULATION CCCCCCCCC

       INTEGER  RUN,EP,BC,EPCNT(EPDIM),BCCNT(BCDIM),SCREEN
       INTEGER  J,K,N,I, SIM, FailSamp, SEP,S,SS,SSS, SET
       INTEGER CONSTRAINTS(3)
```

```
      INTEGER XBASIC(NL,BCDIM,M)
      REAL  NOISEMULT(NL), OPTIMUM(TRUNS,1)
      REAL  BASIS(BCDIM,NVAR), OPTBASIS(BCDIM,TRUNS)
      REAL  EXTPT(EPDIM, NVAR), OPTEP(EPDIM,TRUNS)
      REAL  TESTU, TESTL, TOL, TEMP, AVE, MIN, MAX, S2
      REAL  OPTDR(DRUNS)
      REAL  DESIGN(DP,NVAR+1),Y(DP,1),BHAT(NVARY,1),VARCOV(NVAR,NVAR)
      REAL  SEXTPT(EPDIM,NVAR), box(STEP,NVAR), SDEV(3)
      REAL  BA_SET(NL,BCDIM,M,NVAR),BA(M,NVAR),CBA(NVAR)
      REAL  CB(M), BMAT(M,M),B_TEST(M)

      COMMON  OPTBASIS,OPTEP,BCCNT,BASIS,EPCNT,EXTPT,XSOL,OBJ
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCCCCCCCCCCCCCC   Definitions   CCCCCCCCCCCCCCCCCCCCCCCC

      CALL UMACH (2, NOUT)
      DATA XLB/NVAR*0.0/
      DATA XUB/NVAR*-1.0E30/
      DATA A/ 1., 2., -1.,
     + 3*1.,
     +  2., -1., 1.,
     + 1.,1.,2./
      DATA B/12., 14., 10./
      DATA OBJTRUE/15.,17.,18.,20./

      CONSTANT = 10.0
      DATA IRTYPE/1, 1, 1/

C DEFINES THE TYPE OF CONSTRAINTS
      DATA CONSTRAINTS /3, 0, 0/

C SAMPLING = 1 <> MONTE CARLO  =2 <> DESIGN
      SAMPLING = 2

C Define standard deviation multipliers
      SDEV(1) = 1.5
      SDEV(2) = 2.75
      SDEV(3) = 4.0

C  SCREEN =1 <> YES, SCREENING,   =0 <> NO SCREENING
      SCREEN = 1

      NOISEMULT(1) = 1.0
      NOISEMULT(2) = 5.0
      NOISEMULT(3) = 9.0
      NOISEMULT(4) = 13.0
      NOISEMULT(5) = 17.0
      TOL = 0.00001


CCCCCCC FULL FACTORIAL WITH 4 VARIABLES  CCCCCCCCCCCCCCCC
      DATA DESIGN/16*1.,
     + -1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,
     + -1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,
```

```
     + 4*-1.,4*1.,4*-1.,4*1.,
     + 8*-1.,8*1./

CCCCCCC HALF FACTORIAL WITH 4 VARIABLES BLOCKING I=1234 CCC
C          DATA DESIGN/8*1,
C      + -1.,1.,1.,-1.,1.,-1.,-1.,1.,
C      + -1.,1.,-1.,1.,-1.,1.,-1.,1.,
C      + -1.,-1.,1.,1.,-1.,-1.,1.,1.,
C      + -1.,-1.,-1.,-1.,1.,1.,1.,1./

CCCCCCC FULL FACTORIAL WITH 5 VARIABLES  CCCCCCCCCCCCCCCC
C          DATA DESIGN/32*1.,
C      + -1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,
C      + -1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,
C      + -1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,
C      + -1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,
C      + 4*-1.,4*1.,4*-1.,4*1.,4*-1.,4*1.,4*-1.,4*1.,
C      + 8*-1.,8*1.,8*-1.,8*1.,
C      + 16*-1.,16*1./

CCCCCCC FULL FACTORIAL WITH 6 VARIABLES  CCCCCCCCCCCCCCCC
C          DATA DESIGN/64*1.,
C      + -1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,
C      + -1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,
C      + -1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,
C      + -1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,-1.,1.,

C      + -1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,
C      + -1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,
C      + -1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,
C      + -1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,-1.,-1.,1.,1.,

C      + 4*-1.,4*1.,4*-1.,4*1.,4*-1.,4*1.,4*-1.,4*1.,
C      + 4*-1.,4*1.,4*-1.,4*1.,4*-1.,4*1.,4*-1.,4*1.,
C      + 8*-1.,8*1.,8*-1.,8*1.,8*-1.,8*1.,8*-1.,8*1.,
C      + 16*-1.,16*1.,16*-1.,16*1.,
C      + 32*-1.,32*1./

C          MODIFIED five level BOX-BENKIN DESIGN FOR 4 VARS
           DATA BOX/-1.,1.,-1.,1., 4*0.0,
     +  -1.,1.,-1.,1., 4*0.0,  -1.,1.,-1.,1., 5*0.0,
     +  -.5,.5,-.5,.5, 4*0.0,
     +  -.5,.5,-.5,.5, 4*0.0,  -.5,.5,-.5,.5, 5*0.0,

     +  -.5,-.5,.5,.5,8*0.0,-1.,1.,-1.,1.,4*0.0,
     +  -1.,1.,-1.,1.,0.0,
     +  -1.,-1.,1.,1.,8*0.0,-.5,.5,-.5,.5,4*0.0,
     +  -.5,.5,-.5,.5,0.0,

     +  4*0.0,-1.,1.,-1.,1.,4*0.0,-.5,-.5,.5,.5,
     +  -.5,-.5,.5,.5,5*0.0,
     +  4*0.0,-.5,.5,-.5,.5,4*0.0,-1.,-1.,1.,1.,
     +  -1.,-1.,1.,1.,5*0.0,
```

```
     +  4*0.0,-.5,-.5,.5,.5,-.5,-.5,.5,.5,8*0.0,
     +  -.5,-.5,.5,.5,0.0,
     +  4*0.0,-1.,-1.,1.,1.,-1.,-1.,1.,1.,8*0.0,
     +  -1.,-1.,1.,1.,0.0/
C       MODIFIED three level BOX-BENKIN DESIGN FOR 5 VARS
C       DATA BOX/-1.,1.,-1.,1., 0.0*8,-1.,1.,-1.,1., 0.0*12,
C     +  -1.,1.,-1.,1., 0.0*4,-1.,1.,-1.,1., 0.0*8,
C
C     +  -1.,-1.,1.,1.,0.0*4,-1.,1.,-1.,1.,0.0*12,
C     +  -1.,1.,-1.,1.,0.0*12,-1.,1.,-1.,1.,0.0*4,
C
C     +  0.0*4,-1.,1.,-1.,1.,0.0*4,-1.,-1.,1.,1.,0.0*8,
C     +  -1.,-1.,1.,1.,0.0*4,-1.,1.,-1.,1.,0.0*12,
C
C     +  0.0*4,-1.,-1.,1.,1.,0.0*8,-1.,-1.,1.,1.,0.0*8,
C     +  -1.,-1.,1.,1.,0.0*8,-1.,-1.,1.,1.,0.0*4,
C
C     +  0.0*8,-1.,-1.,1.,1.,0.0*4,-1.,-1.,1.,1.,0.0*12,
C     +  -1.,-1.,1.,1.,-1.,-1.,1.,1.,0.0*8/


CCCCCCCCCCCC   SOLVE LP FOR TRUE ANSWER CCCCCCCCCCCCCCCCCC

       DO 2 K = 1, NVAR
         C(K) = OBJTRUE(K)
2      CONTINUE

C   TO MAXIMIZE, C MUST BE MULTIPLIED BY -1
       CALL SSCAL (NVAR, -1.0E0, C, 1)

C  SET BOUNDS ON X VARIABLES
       do 3 i = 1,nvar
         xlb(i) = 0.0
         xub(i) =-1.0E30
3      continue

       CALL DLPRS (M,NVAR,A,M,B,B,C,IRTYPE,XLB,XUB,
     +              OPTT,XSOLT,DSOL)

C   AFTER RUN DSOL MUST BE MULTIPLIED BY -1 TO GET TRUE MAX
       CALL SSCAL (M, -1.0E0, DSOL, 1)

C      ADDING THE CONSTANT TERM
       OPTT = -OPTT + CONSTANT

CCCCCCCCCCCCCCCCCC  END OF TRUE ANSWER FOR LP  CCCCCCCCCCCCCCC

CCCCCC  INITIALIZE SEED OF RANDOM NUM GENERATOR  CCCCCCCCCCCCC

       ISEED= 1735927
       CALL RNSET (ISEED)
       OPEN (UNIT = 20, FILE= 'final.out', STATUS ='NEW')

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
```

```
CCCCCCCCCCCCCCCC  Monte Carlo Simulation  CCCCCCCCCCCCCCCCCCCC
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

CCCCCCCCCCCCCCCCCCCC  NOISE LOOP  CCCCCCCCCCCCCCCCCCCCCCCCCCCC

       DO 900 N = 1,NL
       write(*,*) 'Noise level ',N
       FailSamp = 0
       EP = 1
       BC = 1
       DO 4 K = 1,EPDIM
         EPCNT(K) = 0
4      CONTINUE

       DO 6 K = 1,BCDIM
         BCCNT(K) = 0
6      CONTINUE
CCCCCCCCCCCCCCCCCC OBJECTIVE FUNCTION GENERATION LOOP CCCCCCCCCC
       DO 500 SIM=1,DRUNS
        DO 20 J=1,DP
         Y(J,1) = 0.0
          DO 10 K=2,NVAR+1
              Y(J,1)= Y(J,1)+DESIGN(J,K)*OBJTRUE(K-1)
10       CONTINUE

        Y(J,1)=Y(J,1)+NOISEMULT(N)*RNNOF()+CONSTANT

20     CONTINUE

       CALL REGRESSION (DESIGN,Y,BHAT,VARCOV)

CCCCCCCCCCCCCC SINGLE SAMPLE WITH NO NOISE CCCCCCCCCCCCCCCCCCCC

       DO 80 K = 1, NVAR
         C(K) = BHAT(K+1,1)
80     CONTINUE

C  TO MAXIMIZE, C MUST BE MULTIPLIED BY -1
       CALL SSCAL (NVAR, -1.0E0, C, 1)

C  Solve LP
       CALL DLPRS (M,NVAR,A,M,B,B,C,IRTYPE,XLB,XUB,
     +             OBJ,XSOL,DSOL)

C  AFTER RUN DSOL MUST BE MULTIPLIED BY -1 TO GET TRUE MAX
       CALL SSCAL (M, -1.0E0, DSOL, 1)

C  ADDING THE CONSTANT TERM
       OBJ = -OBJ + Bhat(1,1)


CCCCCCCCCCC END SINGLE SAMPLE WITH NO NOISE CCCCCCCCCCCCCCCCCC
```

```
C OBTAIN THE CHOLESKY FACTORIZATION (GEN OF MULTIVARIATE DIST)
       CALL CHFAC (NVAR,VARCOV,NVAR,0.00001,IRANK,RSIG,NVAR)

       CALL RNMVN (RUNS, NVAR, RSIG, NVAR, R, RUNS)

C   RNMVN (NR, K, RSIG, LDRSIG, R, LDR)
C   NR - # RANDOM MULTIVARIATE NORMAL VECTORS TO GENERATE (INPUT)
C   K - (EQ NVAR) LENGTH OF THE MULTIVARIATE NORMAL VECTOR (INPUT)
C   RSIG - UPPER TRIANG MATRIX, K BY K, CONTAINING THE CHOLESKY FACTOR
FOR THE
C      VARIANCE-COVARIANCE MATRIX (INPUT)
C   LDRSIG -(RUNS) LEADING DIM OF RSIG EXACTLY AS SPECIFIED IN THE
CALLING
C     PROGRAM
C   (INPUT)
C   R - NR BY K MATRIX CONTAINING THE RANDOM VECTOR IN ITS ROWS
C   LDR (RUNS) - LEADING DIM OF R EXACTLY AS SPECIFIED IN THE DIMENSION
C     STATEMENT OF THE CALLING PROGRAM (INPUT)

CCCCCCCCCCCCCCCC  OBJECTIVE FUNCTION SAMPLING LOOP CCCCCCCCCCC

       OPTDR(SIM)=0.0
       SEP = 1
       SSS =0
       SET = 0
       DO 390 RUN = 1, RUNS

       DO 100 K = 1, NVAR
        IF (SAMPLING.EQ.1) THEN
         C(K) = BHAT(K+1,1) + R(RUN,K)
        ELSE
         SS = RUN - SSS*STEP
         S = SSS +1
         SSS=INT(RUN/STEP)
         C(K)=BHAT(K+1,1)+
     +  BOX(SS,K)*SDEV(S)*SQRT(VARCOV(K,K))

        ENDIF
100    CONTINUE

       IF (SCREEN.EQ.1) GOTO 120

C  TO MAXIMIZE, C MUST BE MULTIPLIED BY -1
       CALL SSCAL (NVAR, -1.0E0, C, 1)

C  Solve LP
       CALL DLPRS (M,NVAR,A,M,B,B,C,IRTYPE,XLB,XUB,
     +             OBJ,XSOL,DSOL)

C  AFTER RUN DSOL MUST BE MULTIPLIED BY -1 TO GET TRUE MAX
       CALL SSCAL (M, -1.0E0, DSOL, 1)

C  ADDING THE CONSTANT TERM
       OBJ = -OBJ + Bhat(1,1)
```

```
C   DLPRS (M, NVAR, A, LDA, BL, BU, C, IRTYPE, XLB, XUB, OBJ, XSOL,
DSOL)
C   M - # OF CONSTRAINTS (INPUT)
C   NVAR - # OF VARIABLES (INPUT)
C   A - MATRIX OF DIM m BY NVAR CONTAINING THE COEFFICIENTS OF THE M
CONST.
C  (INPUT)
C   LDA - LEADING DIM OF A EXACTLY AS SPECIFIED IN THE DIM STATEMENT
(INPUT)
C   BL, BU - UPPER & LOWER BOUNDS VECTOR ENGTH M (INPUT)
C   C - VECTOR LENGTH NVAR = COEFF. OF OBJ FUNCT (INPUT)
C   IRTYPE - VECTOR LENGTH M = TYPE OF CONSTRAINTS EXCLUSIVE OF SIMPLE
BOUNDS
C   WHERE IRTYPE(I)= 0,1,2,3 INDICATES .EQ., .LE., .GE., AND RANGE
CONSTRAINTS
C   RESPECTIVILY
C   XLB,XUB - VECTORS LENGTH NVAR LOWER & UPPER BOUNDS OF VARIABLES
C    IF NO BOUNDS THEN SET XLB = 1.0E30, OR XUB = -1.0E30
C    DEPENDING WHICH IS UNBND
C   OBJ - VALUE OF OBJECTIVE FUNCTION (OUTPUT)
C   XSOL - VECTOR LENGTH NVAR = PRIMAL SLOUTION (OUPUT)
C   DSOL - VECTOR LENGTH M = DUAL SOLUTION (OUTPUT)

       GOTO 155

C selection procedure to follow

120      IF (RUN.GT.1) THEN
           DO 135 J= 1,SET

C DEFINE BASIC COEFFICIENTS
            DO 121 I=1,M
             TEMP = XBASIC(N,J,I)
             IF (TEMP.EQ.0) THEN
               CB(I) = 0.0
             ELSE
               CB(I) = C(XBASIC(N,J,I))
             ENDIF
121         CONTINUE

C CALCULATE Cb*Binv*A VECTOR GIVEN BA = Binv*A
         DO 124 K= 1, NVAR
           CBA(K) = 0.0
           DO 122 I = 1, M

             CBA(K)=CBA(K)+CB(I)*BA_SET(N,J,I,K)

122        CONTINUE
124      CONTINUE

C OPTIMALITY TEST - BRANCH IF NOT OPTIMAL
             DO 130 I= 1,NVAR
               IF(C(I)-CBA(I).GT.tol) GOTO 134
```

```
130         CONTINUE

C BASIS ALREADY SAMPLED - OPTIMAL CONDITIONS MET, DON'T SAMPLE
C Branch and sample objective function agian
        GOTO 390

134     temp=0
135     CONTINUE

      ELSE
        SET = 0
      ENDIF

      SET = SET +1

C  TO MAXIMIZE, C MUST BE MULTIPLIED BY -1
      CALL SSCAL (NVAR, -1.0E0, C, 1)

C  Solve LP
      CALL DLPRS (M,NVAR,A,M,B,B,C,IRTYPE,XLB,XUB,
     +            OBJ,XSOL,DSOL)

C  AFTER RUN DSOL MUST BE MULTIPLIED BY -1 TO GET TRUE MAX
      CALL SSCAL (M, -1.0E0, DSOL, 1)

C  ADDING THE CONSTANT TERM
      OBJ = -OBJ + Bhat(1,1)

      COUNT = 0
      DO 137 I = 1, NVAR
        IF(XSOL(I).NE.0.0) THEN
          COUNT = COUNT+1
          XBASIC(N,SET,COUNT) = I
         DO 136 J = 1, M
           BMAT(J,COUNT) = A(J,I)
136     CONTINUE
        ENDIF
137     CONTINUE

      IF(COUNT.GT.M) WRITE (*,*) 'ERROR COUNT > M'
C
CCC  FOLLWING IS EXECUTED WHEN A BASIC VARIABLE IS NOT A DECISION
VARIABLE CCC
C
      IF (COUNT.LT.M) THEN
       DO 142 I = COUNT+1, M
         XBASIC(N,SET,I)= 0.0
         DO 140 J = 1, M
          BMAT(J,I) = 0.0
140      CONTINUE
142     CONTINUE

       DO 148 I= 1, M
         B_TEST(I) = 0.0
```

```
          DO  146 J = 1, NVAR
            B_TEST(I)=B_TEST(I)+XSOL(J)*A(I,J)
146       CONTINUE
148      CONTINUE

       DO 150 I = 1, M
         IF ((B_TEST(I)-B(I).NE.0.0).AND.((DSOL(I).LE.TOL)
     +  .AND.(DSOL(I).GE.-TOL))) THEN
           COUNT=COUNT+1
            BMAT(I,COUNT) = 1
         ENDIF
150      CONTINUE
       ENDIF

       IF (COUNT.GT.M) WRITE (*,*) 'ERROR: TOO MANY B coln DEFINED'

C  INVERTING B
       CALL LINRG(M,BMAT,M,BMAT,M)

C  FINDING Binv*A
       CALL MRRRR(M,M,BMAT,M,M,NVAR,A,M,M,NVAR,BA,M)

CCCCCCCC   END OF FINDING B WHEN BASIC VAR IS NOT A DECISION VAR
CCCCCCCCCCCC

       DO 153  J=1,M
         DO 151 I= 1,NVAR
           BA_SET(N,SET,J,I) = BA(J,I)
151      CONTINUE

153    CONTINUE

C  DEFINE RUNNING STATISTICS
155    OPTDR(SIM) = OPTDR(SIM) + OBJ
       OPTIMUM(RUN+ RUNS*(SIM-1),1) = OBJ

CCCCCCCC  DEFINE FOR PER SAMPLE TESTING  CCCCCCCCCCCCCCCCCCCC
       IF (RUN.EQ.1) THEN
         DO  157 K = 1, NVAR
          SEXTPT(1,K) = XSOL(K)
157      CONTINUE
       ENDIF

C  Define first Extreme pt & Decision Set for each run

       IF ((RUN.EQ.1).AND.(SIM.EQ.1)) THEN
         BCCNT(1) = 1
         OPTBASIS(1,1) = OBJ
         OPTEP(1,1) = OBJ
         EPCNT(1) = 1
         DO  158 K = 1, NVAR
          EXTPT(1,K) = XSOL(K)
          BASIS(1,K) = XSOL(K)
158      CONTINUE
```

```
          GO TO 390
       ENDIF

CCCCCCCCCCCCCC TEST COMPARE EXTREME POINTS PER SAMPLE CCCCCC
       J = SEP
160    IF (J.GT.O) THEN
          DO 170 K = 1, NVAR
           TESTL = XSOL(K) - TOL
           TESTU = XSOL(K) + TOL
           IF ((SEXTPT(J,K).LT.TESTL).OR.(SEXTPT(J,K).GT.TESTU)) THEN
             J= J-1
             IF (J.EQ.0) THEN
              GO TO 175
             ELSE
             GO TO 160
             ENDIF
           ENDIF
170       CONTINUE
        GO TO 199

C  DEFINE A NEW EXTREME POINT FOR THIS RUN ONLY

175     SEP = SEP +1
        DO 180 K = 1,NVAR
          SEXTPT(SEP,K) = XSOL(K)
180     CONTINUE
        ENDIF

CCCCCCCCCCCCCCC  TEST TO COMPARE EXTREME POINTS CCCCCCCCCCCC
199    J = EP
200    IF (J.GT.O) THEN
          DO 250 K = 1, NVAR
           TESTL = XSOL(K) - TOL
           TESTU = XSOL(K) + TOL
           IF ((EXTPT(J,K).LT.TESTL).OR.(EXTPT(J,K).GT.TESTU)) THEN
             J= J-1
             IF (J.EQ.0) THEN
              GO TO 275
             ELSE
             GO TO 200
             ENDIF
           ENDIF
250       CONTINUE
        CALL ASSIGNEP(J)
        J = J -1
        GO TO 290

C  DEFINE A NEW EXTREME POINT

275     EP = EP +1
        CALL ASSIGNEP(EP)
        ENDIF
```

```
CCCCCCCC  TEST TO COMPARE DECIOSON VAR SET CHANGES  CCCCCCCC

290    J = BC
300     DO 350 K = 1, NVAR
            IF ((((BASIS(J,K).LE.TOL).AND.(BASIS(J,K).GE.-TOL))
     +    .AND.((XSOL(K).GE.TOL). OR.(XSOL(K).LE.-TOL)))
     +    .OR.(((BASIS(J,K).GE.TOL). OR.(BASIS(J,K).LE.-TOL)))
     +    .AND.((XSOL(K).LE.TOL).AND.(XSOL(K).GE.-TOL))) THEN
             GO TO 355
            ENDIF

350       CONTINUE
          CALL ASSIGNBC (J)
          GO TO 390

355       IF (J.LE.1) THEN
          BC = BC+1
          CALL ASSIGNBC (BC)
          ELSE
           J = J -1
           GO TO 300
          ENDIF

390      CONTINUE

CCCCCCCC TEST IF TRUE EXTREME POINT WAS SAMPLED  CCCCCCCCCCC
       J = SEP
400    IF (J.GT.0) THEN
          DO 470 K = 1, NVAR
           TESTL = XSOLT(K) - TOL
           TESTU = XSOLT(K) + TOL
           IF ((SEXTPT(J,K).LT.TESTL).OR.(SEXTPT(J,K).GT.TESTU)) THEN
             J = J-1
             IF (J.EQ.0) THEN
               GO TO 480
             ELSE
               GO TO 400
             ENDIF
           ENDIF
470       CONTINUE
        GO TO 500

C  DEFINE A NEW EXTREME POINT FOR THIS RUN ONLY

480     FailSamp = FailSamp + 1
        ENDIF

500    CONTINUE


CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCCCCCCCCCCCC  END IF FOR MONTE CARLO SIM CCCCCCCCCCCCCCCC
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCCCCCCCCCC CALCULATE RESULTS AND PRINT OUTPUT CCCCCCCCCCCC
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

       CALL UMACH(-2,20)
       IF ((XLB(1).NE. 0.).OR.(XLB(2).NE. 0.)) THEN
         WRITE(20,*)'******* WARNING  WARNING ************'
         WRITE (20,*) '  LOWER BOUNDS ON X CORRUPT'
         WRITE (20,*) 'XLB = ',XLB
        ENDIF

       WRITE (20,*) 'NOISE MULTIPLIER (SD) FOR NORMAL NOISE: ',
     + NOISEMULT(N)
       TEMP = NOISEMULT(N)*(1/real(DP))**.5
       WRITE (20,*) 'STANDARD ERROR IS :',TEMP
       if (screen.eq.1) WRITE(20,*) 'This is a screening run'
       if (sampling.eq.2) write(20,*) '   Using a Design to sample'
       WRITE (20,*) '# Times failed to sample true extreme pt',
     + FailSamp
       write(20,*) 'Standard deviation mult set =',SDEV
       IF (FailSamp.GT.0) then
         TEMP = 100*REAL(FailSamp)/real(DRUNS)
         WRITE (20,'(A, F7.3)') '% failures overall: ', TEMP
       ENDIF
       WRITE (20,*)'Number of Objective Function Samples: ',DRUNS
       WRITE (20,*)'Number of Runs per Obj Function: ', RUNS
       WRITE (20,*)'Total Number of Points Tested: ',TRUNS
       WRITE (20,*)'The True Objective Function: '
       WRITE (20,*) 'const',CONSTANT,'+',OBJTRUE
       WRITE (20,*) 'Sample Generated Objective Function'
       WRITE (20,*) BHAT
       CALL WRRRN ('Constraint Matrix',LDA,NVAR,A,LDA,0)
       WRITE (20,*) ''
       WRITE (20,*) 'The RHS is: ', B
       WRITE (20,*) '*True Optimal Answer: ', OPTT
       WRITE (20,*) '*True optimal Extreme Point:'
       WRITE (20,*)  XSOLT
       CALL WRRRN ('Design Matrix',DP,NVAR+1,DESIGN,DP,0)
       WRITE (20,*) 'Sample response varaible Y:'
       WRITE (20,*) Y
       CALL WRRRN ('Sample Variance-Covariance Matrix',
     + NVAR,NVAR,VARCOV,NVAR,0)
       CALL WRRRN ('Sample Cholesky Factorization Matrix',
     + NVAR,NVAR,RSIG,NVAR,1)
       if (sampling.eq.2) then
        CALL WRRRN ('SAMPLING DESING (BOX-BEHNKEN)',
     +  step,NVAR,BOX,step,0)
       endif
       WRITE (20,*) ''
       WRITE (20,*)
     + '!!!!!!!!!!!!!!!!!! DECISION VARIABLES !!!!!!!!!!!!!!!!!'

       WRITE (20,*) '# OF DECISION VARIABLE SET CHANGES : ',BC
       WRITE (20,*) ''
```

```
      DO 680 K = 1, BC
        WRITE (20,*) '  '
        WRITE (20,*) 'Decision var set # ',K

        DO 605 J = 1, NVAR
          WRITE (20,'(2X,f10.4)') BASIS(K,J)
605     CONTINUE

        WRITE (20,*) '# of occurances of this basis: ',BCCNT(K)
        if (sampling.eq.1) then
           TEMP = (100* REAL(BCCNT(K)))/TRUNS
        else
           TEMP = (100*REAL(BCCNT(K)))/DRUNS
        ENDIF
        WRITE (20,'(A, F7.3)') '% of overall occurance is: ', TEMP
        AVE = 0.0

        DO 608 J = 1, BCCNT(K)
          AVE = AVE + OPTBASIS(K,J)
608     CONTINUE

        AVE = AVE/REAL(BCCNT(K))
        WRITE (20,'(A, F10.4)') 'AVE. OPTIMUM IS: ',AVE
        WRITE (20,'(A,F10.4)')'Bias opt est (Ave Opt-True Opt): ',
     +  AVE - OPTT
        S2 = 0.0
        MIN = OPTBASIS(K,1)
        MAX = MIN

        DO 620 J = 1, BCCNT(K)
          S2 = S2+ (AVE - OPTBASIS(K,J))**2

          IF (MAX.LT.OPTBASIS(K,J)) THEN
            MAX = OPTBASIS(K,J)
          ENDIF

          IF (MIN.GT.OPTBASIS(K,J)) THEN
            MIN = OPTBASIS (K,J)
         ENDIF

620     CONTINUE

         S2 = S2/BCCNT(K)
         WRITE (20,'(A, F13.4)') 'The population variance is: ', S2
         WRITE (20,'(A, F13.4)') 'The maximum value is: ', max
         WRITE (20,'(A, F13.4)') 'The minimum value is: ', min

680   CONTINUE

      WRITE (20,*) ''
      WRITE (20,*) ''

      WRITE (20,*)
```

```
     + '!!!!!!!!!!!!!!!!!!!  THE EXTREME POINT !!!!!!!!!!!!!!!!!!!'
       WRITE (20,*) 'NUMBER OF EXT POINTS VISITED IS: ', EP
       DO 770 K = 1, EP
         WRITE (20,*) '  '
         WRITE (20,*) 'Extreme Point # ', K
         DO 715 J = 1, NVAR
           WRITE (20,'(2X,f10.4)') EXTPT(K, J)
715      CONTINUE

         WRITE (20,*) '# OF EXT PT VISITS ARE: ',EPCNT(K)
         if (sampling.eq.1) then
            TEMP = (100* REAL(EPCNT(K)))/TRUNS
         else
            TEMP = (100*REAL(EPCNT(K)))/DRUNS
         ENDIF

         WRITE (20,'(A, F7.3)') '% OF OVERALL IS: ', TEMP
         AVE = 0.0

         DO 718 J = 1, EPCNT(K)
           AVE = AVE + OPTEP(K,J)
718      CONTINUE

        AVE = AVE/REAL(EPCNT(K))
        WRITE (20,'(A, F10.4)') 'AVE. OPTIMUM IS: ',AVE
        WRITE (20,'(A, F10.4)')'Bias opt est (Ave Opt-True Opt): ',
     +  AVE - OPTT

C  CALCULATE TRUE Z* BASED ON THIS EXTREME POINT
        WRITE(20,*) ''
        TEMP = 0.0
        DO 719 J=1,NVAR
           TEMP = TEMP + OBJTRUE(J)*BASIS(K,J)
719     CONTINUE
        TEMP = TEMP + CONSTANT
        WRITE (20,'(A, F10.4)') 'TRUE Z* WITH TRUE C IS: ',TEMP
        WRITE (20,'(A, F10.4)') 'TRUE BIAS (Z* - Z optimal): ',
     +  TEMP - OPTT
        WRITE (20,'(A, F10.4)')
     + 'Difference between expected optimal and true',TEMP-AVE


C  CALCULATE VARIANCE, MINIMUM & MAXIMUM
        S2 = 0.0
        MIN = OPTEP(K,1)
        MAX = MIN

        DO 741 J = 1, EPCNT(K)
          S2 = S2+ (AVE - OPTEP(K,J))**2

          IF (MAX.LT.OPTEP(K,J)) THEN
            MAX = OPTEP(K,J)
          ENDIF
          IF (MIN.GT.OPTEP(K,J)) THEN
```

```
           MIN = OPTEP (K,J)
          ENDIF

741      CONTINUE

          S2 = S2/EPCNT(K)
          WRITE (20,'(A, F13.4)') 'The population variance is: ', S2
          WRITE (20,'(A, F13.4)') 'The maximum value is: ', max
          WRITE (20,'(A, F13.4)') 'The minimum value is: ', min
770     CONTINUE

         WRITE (20,*) ''
         WRITE (20,*) ''

         WRITE (20,*)
       + '!!!!!!!!!!!!!!!!!!!! OVERALL RESULTS !!!!!!!!!!!!!!!!!!!!!!'
         WRITE (20,*) ''

          AVE = 0.0
          DO 800 J = 1, TRUNS
            AVE = AVE + OPTIMUM(J,1)
800       CONTINUE

         AVE = AVE/TRUNS
         WRITE (20,'(A, F10.4)') 'Overall Mean Optimum is: ',AVE
         WRITE (20,'(A, F10.4)')'Overall Bias (Ave Opt-True Opt): ',
       + AVE - OPTT
         S2 = 0.0
         MIN = OPTIMUM(1,1)
         MAX = MIN

         DO 820 J = 1, TRUNS
           S2 = S2+ (AVE - OPTIMUM(J,1))**2

           IF (MAX.LT.OPTIMUM(J,1)) THEN
             MAX = OPTIMUM(J,1)
           ENDIF
           IF (MIN.GT.OPTIMUM(J,1)) THEN
             MIN = OPTIMUM(J,1)
           ENDIF

820      CONTINUE
         S2 = S2/(TRUNS-1)
         WRITE (20,'(A, F13.4)') 'The overall sample variance is: ', S2
         WRITE (20,'(A, F13.4)') 'The overall maximum value is: ', max
         WRITE (20,'(A, F13.4)') 'The overall minimum value is: ', min
         WRITE (20,*) ''
C
        WRITE (20,*)''
        WRITE(20,*)'AVE OPTIMAL PER OBJECTIVE FUNCTION'
        DO 855 J = 1, DRUNS
          OPTDR(J) = OPTDR(J)/RUNS
855     CONTINUE
```

```
      AVE = 0.0
      DO 856 J = 1, DRUNS
          AVE = AVE + OPTDR(J)
856   CONTINUE

      AVE = AVE/DRUNS
      WRITE(20,'(A, F10.4)')'Mean of Mean Opt per obj funct: ',AVE

      S2 = 0.0
      MIN = OPTDR(1)
      MAX = MIN

      DO 857 J = 1, DRUNS
        S2 = S2+ (AVE - OPTDR(J))**2

        IF (MAX.LT.OPTDR(J)) THEN
          MAX = OPTDR(J)
        ENDIF
        IF (MIN.GT.OPTDR(J)) THEN
          MIN = OPTDR(J)
        ENDIF

857    CONTINUE
       S2 = S2/(DRUNS-1)
       WRITE (20,'(A, F13.4)') 'The sample var(mean opt): ', S2
       WRITE (20,'(A, F13.4)') 'The maximum value is: ', max
       WRITE (20,'(A, F13.4)') 'The minimum value is: ', min
       WRITE (20,*) ''

      DO 860 J = 1,15
        WRITE (20,*) ''
860   CONTINUE

900   CONTINUE

      CLOSE (20)


      END

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCCCCCCCCCCCCCCCCCC  END PROGRAM  CCCCCCCCCCCCCCCCCCCCCCCCCC
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      SUBROUTINE ASSIGNBC (TEMP)

      INTEGER TEMP, KK, NVAR, BCDIM, EPDIM,RUNS
      PARAMETER (NVAR=4,BCDIM=20, EPDIM=20,RUNS=149)
      INTEGER DP, DRUNS, TRUNS, NVARY
      PARAMETER (DP=16,DRUNS=1000,TRUNS=RUNS*DRUNS,NVARY=NVAR+1)

      INTEGER EPCNT(EPDIM), BCCNT(BCDIM)
      REAL  OPTBASIS(BCDIM,TRUNS)
      REAL  OPTEP(EPDIM,TRUNS),BASIS(BCDIM,NVAR)
```

```
      REAL  XSOL(NVAR),OBJ, EXTPT(EPDIM,NVAR)
      COMMON OPTBASIS,OPTEP,BCCNT,BASIS,EPCNT,EXTPT,XSOL,OBJ

C  COUNT OCCURANCES OF EACH DECISION SET CHANGE
      BCCNT(TEMP) = BCCNT(TEMP) + 1

C  DEFINE DECISION SET
      OPTBASIS(TEMP,BCCNT(TEMP)) = OBJ
      DO  1010 KK = 1, NVAR
        BASIS(TEMP,KK) = XSOL(KK)
1010  CONTINUE

      RETURN
      END

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      SUBROUTINE ASSIGNEP (TEMP)

      INTEGER TEMP, KK, NVAR, BCDIM, EPDIM,RUNS
      PARAMETER (NVAR=4,BCDIM=20, EPDIM=20,RUNS=149)
      INTEGER DP, DRUNS, TRUNS, NVARY
      PARAMETER (DP=16,DRUNS=1000,TRUNS=RUNS*DRUNS,NVARY=NVAR+1)

      INTEGER EPCNT(EPDIM), BCCNT(BCDIM)
      REAL  OPTBASIS(BCDIM,TRUNS)
      REAL  OPTEP(EPDIM,TRUNS),BASIS(BCDIM,NVAR)
      REAL  XSOL(NVAR),OBJ, EXTPT(EPDIM,NVAR)
      COMMON OPTBASIS,OPTEP,BCCNT,BASIS,EPCNT,EXTPT,XSOL,OBJ

C  COUNT OCCURANCES OF EACH EXTREME POINT
      EPCNT(TEMP) = EPCNT(TEMP) + 1

C  DEFINE EXTREME POINT
      OPTEP(TEMP,EPCNT(TEMP)) = OBJ
      DO  1020 KK = 1, NVAR
        EXTPT(TEMP,KK) =  XSOL(KK)
1020  CONTINUE

      RETURN
      END

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      SUBROUTINE REGRESSION(X,Y,BHAT,VARCOV)

      INTEGER NVAR, NVARY,DP,KK, JJ
      PARAMETER (NVAR=4,NVARY=NVAR+1,DP=16)
      REAL X(DP,NVARY),Y(DP,1),BHAT(NVARY,1),VARCOV(NVAR,NVAR)
      REAL XX(NVARY,NVARY),C(NVARY,1),INV(NVARY,NVARY)
      REAL MSE, EY(DP)

      EXTERNAL MXTXF,LSGRR,MXTYF,MRRRR
```

```
C COMPUTE (X'X)
      CALL MXTXF(DP,NVARY,X,DP,NVARY,XX,NVARY)

C INVERT THE MATRIX
      CALL LSGRR(NVARY,NVARY,XX,NVARY,0.00001,2,INV,NVARY)

C COMPUTE X'Y
      CALL MXTYF(DP,NVARY,X,DP,DP,1,Y,DP,NVARY,1,C,NVARY)
      CALL MRRRR(NVARY,NVARY,INV,NVARY,NVARY,1,
     + C,NVARY,NVARY,1,BHAT,NVARY)

C  CALCULATE MSE
      MSE = 0.0

      DO 1105 KK = 1,DP
        EY(KK) = 0.0
        DO 1104 JJ = 1, NVARY
          EY(KK)=EY(KK)+X(KK,JJ)*BHAT(JJ,1)
1104    CONTINUE
1105  CONTINUE

      DO 1107 KK= 1, DP
        MSE = MSE + (Y(KK,1)-EY(KK))**2
1107  CONTINUE
      MSE = MSE/( DP - NVARY)

C  CALCULATE VARIANCE-COVARIANCE MATRIX
      DO 1110 K=2,NVARY

        DO 1100 J=2,NVARY
          VARCOV(J-1,K-1)=INV(J,K)*MSE
1100    CONTINUE

1110  CONTINUE

      RETURN
      END

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
```

Appendix E: Sample Computer Program Output

NOISE MULTIPLIER (SD) FOR NORMAL NOISE: 1.00000
STANDARD ERROR IS : 0.250000
This is a screening run
Using a Design to sample
Times failed to sample true extreme pt 0
Standard deviation mult set = 1.50000 2.75000
4.00000
Number of Objective Function Samples: 1000
Number of Runs per Obj Function: 149
Total Number of Points Tested: 149000
The True Objective Function:
const 10.00000+ 15.0000 17.0000 18.0000
20.0000
Sample Generated Objective Function
9.53095 15.0707 17.1053 17.7075 19.9385

Constraint Matrix

	1	2	3	4
1	1	1	2	1
2	2	1	-1	1
3	-1	1	1	2

The RHS is: 12.0000 14.0000 10.00000
*True Optimal Answer: 216.000
*True optimal Extreme Point:
2.00000 8.00000 0. 2.00000

Design Matrix

	1	2	3	4	5
1	1	-1	-1	-1	-1
2	1	1	-1	-1	-1
3	1	-1	1	-1	-1
4	1	1	1	-1	-1
5	1	-1	-1	1	-1
6	1	1	-1	1	-1
7	1	-1	1	1	-1
8	1	1	1	1	-1
9	1	-1	-1	-1	1
10	1	1	-1	-1	1
11	1	-1	1	-1	1
12	1	1	1	-1	1
13	1	-1	-1	1	1
14	1	1	-1	1	1
15	1	-1	1	1	1
16	1	1	1	1	1

Sample response varaible Y:
-59.8876 -28.8743 -25.9255 4.72923 -25.1038
4.60987 9.42563
37.7662 -20.6727 9.03708 12.6977 43.4840
14.8402 45.4566
50.3080 80.6046

Sample Variance-Covariance Matrix
1 2 3 4

1	0.05979	0.00000	0.00000	0.00000
2	0.00000	0.05979	0.00000	0.00000
3	0.00000	0.00000	0.05979	0.00000
4	0.00000	0.00000	0.00000	0.05979

Sample Cholesky Factorization Matrix

	1	2	3	4
1	0.2445	0.0000	0.0000	0.0000
2		0.2445	0.0000	0.0000
3			0.2445	0.0000
4				0.2445

SAMPLING DESING (BOX-BEHNKEN)

	1	2	3	4
1	-1.0	-0.5	0.0	0.0
2	1.0	-0.5	0.0	0.0
3	-1.0	0.5	0.0	0.0
4	1.0	0.5	0.0	0.0
5	0.0	0.0	-1.0	-0.5
6	0.0	0.0	1.0	-0.5
7	0.0	0.0	-1.0	0.5
8	0.0	0.0	1.0	0.5
9	-1.0	0.0	0.0	-0.5
10	1.0	0.0	0.0	-0.5
11	-1.0	0.0	0.0	0.5
12	1.0	0.0	0.0	0.5
13	0.0	-1.0	-0.5	0.0
14	0.0	1.0	-0.5	0.0
15	0.0	-1.0	0.5	0.0
16	0.0	1.0	0.5	0.0
17	-1.0	0.0	-0.5	0.0
18	1.0	0.0	-0.5	0.0
19	-1.0	0.0	0.5	0.0
20	1.0	0.0	0.5	0.0
21	0.0	-1.0	0.0	-0.5
22	0.0	1.0	0.0	-0.5
23	0.0	-1.0	0.0	0.5
24	0.0	1.0	0.0	0.5
25	0.0	0.0	0.0	0.0
26	-0.5	-1.0	0.0	0.0
27	0.5	-1.0	0.0	0.0
28	-0.5	1.0	0.0	0.0
29	0.5	1.0	0.0	0.0
30	0.0	0.0	-0.5	-1.0
31	0.0	0.0	0.5	-1.0
32	0.0	0.0	-0.5	1.0
33	0.0	0.0	0.5	1.0
34	-0.5	0.0	0.0	-1.0
35	0.5	0.0	0.0	-1.0
36	-0.5	0.0	0.0	1.0
37	0.5	0.0	0.0	1.0
38	0.0	-0.5	-1.0	0.0
39	0.0	0.5	-1.0	0.0
40	0.0	-0.5	1.0	0.0

41	0.0	0.5	1.0	0.0
42	-0.5	0.0	-1.0	0.0
43	0.5	0.0	-1.0	0.0
44	-0.5	0.0	1.0	0.0
45	0.5	0.0	1.0	0.0
46	0.0	-0.5	0.0	-1.0
47	0.0	0.5	0.0	-1.0
48	0.0	-0.5	0.0	1.0
49	0.0	0.5	0.0	1.0
50	0.0	0.0	0.0	0.0

```
!!!!!!!!!!!!!!!!!! DECISION VARIABLES !!!!!!!!!!!!!!!!!!
# OF DECISION VARIABLE SET CHANGES :   2


Decision var set #   1
      4.0000
      0.0000
      0.6667
      6.6667
# of occurances of this basis:   1003
% of overall occurance is: 100.300
AVE. OPTIMUM IS:   215.0843
Bias opt est (Ave Opt-True Opt):    -0.9157
The population variance is:        3.7599
The maximum value is:      220.5652
The minimum value is:      209.3271

Decision var set #   2
      2.0000
      8.0000
      0.0000
      2.0000
# of occurances of this basis:   1000
% of overall occurance is: 100.000
AVE. OPTIMUM IS:   215.6013
Bias opt est (Ave Opt-True Opt):    -0.3987
The population variance is:        3.8427
The maximum value is:      221.0208
The minimum value is:      209.2253


!!!!!!!!!!!!!!!!!!!!  THE EXTREME POINT !!!!!!!!!!!!!!!!!!!!
NUMBER OF EXT POINTS VISITED IS:   2

Extreme Point #   1
      4.0000
      0.0000
      0.6667
      6.6667
# OF EXT PT VISITS ARE:   1003
% OF OVERALL IS: 100.300
AVE. OPTIMUM IS:   215.0843
Bias opt est (Ave Opt-True Opt):    -0.9157
```

```
TRUE Z* WITH TRUE C IS:   215.3333
TRUE BIAS (Z* - Z optimal):     -0.6667
Difference between expected optimal and true    0.2490
The population variance is:         3.7599
The maximum value is:      220.5652
The minimum value is:      209.3271

Extreme Point #   2
      2.0000
      8.0000
      0.0000
      2.0000
# OF EXT PT VISITS ARE:   1000
% OF OVERALL IS: 100.000
AVE. OPTIMUM IS:   215.6013
Bias opt est (Ave Opt-True Opt):     -0.3987

TRUE Z* WITH TRUE C IS:   216.0000
TRUE BIAS (Z* - Z optimal):      0.0000
Difference between expected optimal and true    0.3987
The population variance is:         3.8427
The maximum value is:      221.0208
The minimum value is:      209.2253


!!!!!!!!!!!!!!!!!!!! OVERALL RESULTS !!!!!!!!!!!!!!!!!!!!!!

Overall Mean Optimum is:      2.8948
Overall Bias (Ave Opt-True Opt):  -213.1052
The overall sample variance is:      614.7244
The overall maximum value is:      221.0208
The overall minimum value is:        0.0000


AVE OPTIMAL PER OBJECTIVE FUNCTION
Mean of Mean Opt per obj funct:      2.8948
The sample var(mean opt):       0.0068
The maximum value is:        4.3585
The minimum value is:        2.8164
```

```
NOISE MULTIPLIER (SD) FOR NORMAL NOISE:     5.00000
STANDARD ERROR IS :    1.25000
This is a screening run
   Using a Design to sample
# Times failed to sample true extreme pt  0
Standard deviation mult set =    1.50000    2.75000
4.00000
Number of Objective Function Samples:   1000
Number of Runs per Obj Function:   149
Total Number of Points Tested:   149000
The True Objective Function:
const   10.00000+    15.0000    17.0000    18.0000
20.0000
Sample Generated Objective Function
    8.95856    15.9861    19.6851    19.5614    16.8254

                 Constraint Matrix
            1           2           3           4
 1          1           1           2           1
 2          2           1          -1           1
 3         -1           1           1           2

The RHS is:      12.0000    14.0000   10.00000
*True Optimal Answer:      216.000
*True optimal Extreme Point:
    2.00000    8.00000  0.    2.00000

                     Design Matrix
            1         2         3         4         5
 1          1        -1        -1        -1        -1
 2          1         1        -1        -1        -1
 3          1        -1         1        -1        -1
 4          1         1         1        -1        -1
 5          1        -1        -1         1        -1
 6          1         1        -1         1        -1
 7          1        -1         1         1        -1
 8          1         1         1         1        -1
 9          1        -1        -1        -1         1
10          1         1        -1        -1         1
11          1        -1         1        -1         1
12          1         1         1        -1         1
13          1        -1        -1         1         1
14          1         1        -1         1         1
15          1        -1         1         1         1
16          1         1         1         1         1
Sample response varaible Y:
   -70.7948   -33.0653   -25.8214   10.42741   -18.5379
17.4076    13.7993
    43.6501   -25.0401    1.01661    11.4050    47.0496
3.49555    39.7061
    55.2738    73.3654

         Sample Variance-Covariance Matrix
            1           2           3           4
```

1	2.252	0.000	0.000	0.000
2	0.000	2.252	0.000	0.000
3	0.000	0.000	2.252	0.000
4	0.000	0.000	0.000	2.252

Sample Cholesky Factorization Matrix

	1	2	3	4
1	1.501	0.000	0.000	0.000
2		1.501	0.000	0.000
3			1.501	0.000
4				1.501

SAMPLING DESING (BOX-BEHNKEN)

	1	2	3	4
1	-1.0	-0.5	0.0	0.0
2	1.0	-0.5	0.0	0.0
3	-1.0	0.5	0.0	0.0
4	1.0	0.5	0.0	0.0
5	0.0	0.0	-1.0	-0.5
6	0.0	0.0	1.0	-0.5
7	0.0	0.0	-1.0	0.5
8	0.0	0.0	1.0	0.5
9	-1.0	0.0	0.0	-0.5
10	1.0	0.0	0.0	-0.5
11	-1.0	0.0	0.0	0.5
12	1.0	0.0	0.0	0.5
13	0.0	-1.0	-0.5	0.0
14	0.0	1.0	-0.5	0.0
15	0.0	-1.0	0.5	0.0
16	0.0	1.0	0.5	0.0
17	-1.0	0.0	-0.5	0.0
18	1.0	0.0	-0.5	0.0
19	-1.0	0.0	0.5	0.0
20	1.0	0.0	0.5	0.0
21	0.0	-1.0	0.0	-0.5
22	0.0	1.0	0.0	-0.5
23	0.0	-1.0	0.0	0.5
24	0.0	1.0	0.0	0.5
25	0.0	0.0	0.0	0.0
26	-0.5	-1.0	0.0	0.0
27	0.5	-1.0	0.0	0.0
28	-0.5	1.0	0.0	0.0
29	0.5	1.0	0.0	0.0
30	0.0	0.0	-0.5	-1.0
31	0.0	0.0	0.5	-1.0
32	0.0	0.0	-0.5	1.0
33	0.0	0.0	0.5	1.0
34	-0.5	0.0	0.0	-1.0
35	0.5	0.0	0.0	-1.0
36	-0.5	0.0	0.0	1.0
37	0.5	0.0	0.0	1.0
38	0.0	-0.5	-1.0	0.0
39	0.0	0.5	-1.0	0.0
40	0.0	-0.5	1.0	0.0

41	0.0	0.5	1.0	0.0
42	-0.5	0.0	-1.0	0.0
43	0.5	0.0	-1.0	0.0
44	-0.5	0.0	1.0	0.0
45	0.5	0.0	1.0	0.0
46	0.0	-0.5	0.0	-1.0
47	0.0	0.5	0.0	-1.0
48	0.0	-0.5	0.0	1.0
49	0.0	0.5	0.0	1.0
50	0.0	0.0	0.0	0.0

```
!!!!!!!!!!!!!!!!!! DECISION VARIABLES !!!!!!!!!!!!!!!!!!
# OF DECISION VARIABLE SET CHANGES :   3


Decision var set #   1
      4.0000
      0.0000
      0.6667
      6.6667
# of occurances of this basis:   1002
% of overall occurance is: 100.200
AVE. OPTIMUM IS:   212.5927
Bias opt est (Ave Opt-True Opt):    -3.4073
The population variance is:       91.0682
The maximum value is:       248.6682
The minimum value is:       180.1154

Decision var set #   2
      2.0000
     10.0000
      0.0000
      0.0000
# of occurances of this basis:   290
% of overall occurance is:  29.000
AVE. OPTIMUM IS:   218.1246
Bias opt est (Ave Opt-True Opt):     2.1246
The population variance is:       95.3511
The maximum value is:       245.0287
The minimum value is:       192.3946

Decision var set #   3
      2.0000
      8.0000
      0.0000
      2.0000
# of occurances of this basis:   1000
% of overall occurance is: 100.000
AVE. OPTIMUM IS:   217.7167
Bias opt est (Ave Opt-True Opt):     1.7166
The population variance is:      107.4373
The maximum value is:       252.3086
The minimum value is:       185.5855
```

```
!!!!!!!!!!!!!!!!!!! THE EXTREME POINT !!!!!!!!!!!!!!!!!!!
NUMBER OF EXT POINTS VISITED IS:   4

Extreme Point #   1
      4.0000
      0.0000
      0.6667
      6.6667
# OF EXT PT VISITS ARE:   1002
% OF OVERALL IS: 100.200
AVE. OPTIMUM IS:   212.5927
Bias opt est (Ave Opt-True Opt):    -3.4073

TRUE Z* WITH TRUE C IS:   215.3333
TRUE BIAS (Z* - Z optimal):    -0.6667
Difference between expected optimal and true    2.7406
The population variance is:        91.0682
The maximum value is:      248.6682
The minimum value is:      180.1154

Extreme Point #   2
      1.0000
     11.0000
      0.0000
      0.0000
# OF EXT PT VISITS ARE:   287
% OF OVERALL IS:  28.700
AVE. OPTIMUM IS:   218.0409
Bias opt est (Ave Opt-True Opt):     2.0409

TRUE Z* WITH TRUE C IS:   210.0000
TRUE BIAS (Z* - Z optimal):    -6.0000
Difference between expected optimal and true   -8.0409
The population variance is:        95.5884
The maximum value is:      245.0287
The minimum value is:      192.3946

Extreme Point #   3
      2.0000
      8.0000
      0.0000
      2.0000
# OF EXT PT VISITS ARE:   1000
% OF OVERALL IS: 100.000
AVE. OPTIMUM IS:   217.7167
Bias opt est (Ave Opt-True Opt):     1.7166

TRUE Z* WITH TRUE C IS:   216.0000
TRUE BIAS (Z* - Z optimal):     0.0000
Difference between expected optimal and true   -1.7167
The population variance is:       107.4373
The maximum value is:      252.3086
The minimum value is:      185.5855
```

```
Extreme Point #   4
      2.0000
     10.0000
      0.0000
      0.0000
# OF EXT PT VISITS ARE:   3
% OF OVERALL IS:   0.300
AVE. OPTIMUM IS:   226.1286
Bias opt est (Ave Opt-True Opt):    10.1286

TRUE Z* WITH TRUE C IS:    10.0000
TRUE BIAS (Z* - Z optimal):  -206.0000
Difference between expected optimal and true -216.1286
The population variance is:        7.9169
The maximum value is:      229.8806
The minimum value is:      223.1049


!!!!!!!!!!!!!!!!!!!!! OVERALL RESULTS !!!!!!!!!!!!!!!!!!!!!!

Overall Mean Optimum is:     4.4164
Overall Bias (Ave Opt-True Opt):  -211.5836
The overall sample variance is:      932.7665
The overall maximum value is:      252.3086
The overall minimum value is:        0.0000


AVE OPTIMAL PER OBJECTIVE FUNCTION
Mean of Mean Opt per obj funct:     3.3154
The sample var(mean opt):        0.4727
The maximum value is:       5.9063
The minimum value is:       2.5426
```

NOISE MULTIPLIER (SD) FOR NORMAL NOISE: 9.00000
STANDARD ERROR IS :   2.25000
This is a screening run
Using a Design to sample
Times failed to sample true extreme pt 1
Standard deviation mult set =   1.10000   2.75000
4.00000
% failures overall: 0.100
Number of Objective Function Samples: 1000
Number of Runs per Obj Function: 149
Total Number of Points Tested: 149000
The True Objective Function:
const   10.00000+   15.0000   17.0000   18.0000
20.0000
Sample Generated Objective Function
11.3725   17.5542   15.3801   15.6525   22.4790

Constraint Matrix

	1	2	3	4
1	1	1	2	1
2	2	1	-1	1
3	-1	1	1	2

The RHS is: 12.0000   14.0000   10.00000
*True Optimal Answer: 216.000
*True optimal Extreme Point:
2.00000   8.00000 0.   2.00000

Design Matrix

	1	2	3	4	5
1	1	-1	-1	-1	-1
2	1	1	-1	-1	-1
3	1	-1	1	-1	-1
4	1	1	1	-1	-1
5	1	-1	-1	1	-1
6	1	1	-1	1	-1
7	1	-1	1	1	-1
8	1	1	1	1	-1
9	1	-1	-1	-1	1
10	1	1	-1	-1	1
11	1	-1	1	-1	1
12	1	1	1	-1	1
13	1	-1	-1	1	1
14	1	1	-1	1	1
15	1	-1	1	1	1
16	1	1	1	1	1

Sample response varaible Y:
-37.9410   -35.1582   -41.4563   20.0342   -29.4672
9.90238   2.25696
22.9770   -29.1399   17.1932   17.9181   54.3095
11.6460   60.9039
56.7295   81.2516

Sample Variance-Covariance Matrix

	1	2	3	4
1	9.011	0.000	0.000	0.000
2	0.000	9.011	0.000	0.000
3	0.000	0.000	9.011	0.000
4	0.000	0.000	0.000	9.011

Sample Cholesky Factorization Matrix

	1	2	3	4
1	3.002	0.000	0.000	0.000
2		3.002	0.000	0.000
3			3.002	0.000
4				3.002

SAMPLING DESING (BOX-BEHNKEN)

	1	2	3	4
1	-1.0	-0.5	0.0	0.0
2	1.0	-0.5	0.0	0.0
3	-1.0	0.5	0.0	0.0
4	1.0	0.5	0.0	0.0
5	0.0	0.0	-1.0	-0.5
6	0.0	0.0	1.0	-0.5
7	0.0	0.0	-1.0	0.5
8	0.0	0.0	1.0	0.5
9	-1.0	0.0	0.0	-0.5
10	1.0	0.0	0.0	-0.5
11	-1.0	0.0	0.0	0.5
12	1.0	0.0	0.0	0.5
13	0.0	-1.0	-0.5	0.0
14	0.0	1.0	-0.5	0.0
15	0.0	-1.0	0.5	0.0
16	0.0	1.0	0.5	0.0
17	-1.0	0.0	-0.5	0.0
18	1.0	0.0	-0.5	0.0
19	-1.0	0.0	0.5	0.0
20	1.0	0.0	0.5	0.0
21	0.0	-1.0	0.0	-0.5
22	0.0	1.0	0.0	-0.5
23	0.0	-1.0	0.0	0.5
24	0.0	1.0	0.0	0.5
25	0.0	0.0	0.0	0.0
26	-0.5	-1.0	0.0	0.0
27	0.5	-1.0	0.0	0.0
28	-0.5	1.0	0.0	0.0
29	0.5	1.0	0.0	0.0
30	0.0	0.0	-0.5	-1.0
31	0.0	0.0	0.5	-1.0
32	0.0	0.0	-0.5	1.0
33	0.0	0.0	0.5	1.0
34	-0.5	0.0	0.0	-1.0
35	0.5	0.0	0.0	-1.0
36	-0.5	0.0	0.0	1.0
37	0.5	0.0	0.0	1.0
38	0.0	-0.5	-1.0	0.0
39	0.0	0.5	-1.0	0.0

40	0.0	-0.5	1.0	0.0
41	0.0	0.5	1.0	0.0
42	-0.5	0.0	-1.0	0.0
43	0.5	0.0	-1.0	0.0
44	-0.5	0.0	1.0	0.0
45	0.5	0.0	1.0	0.0
46	0.0	-0.5	0.0	-1.0
47	0.0	0.5	0.0	-1.0
48	0.0	-0.5	0.0	1.0
49	0.0	0.5	0.0	1.0
50	0.0	0.0	0.0	0.0

```
!!!!!!!!!!!!!!!!!! DECISION VARIABLES !!!!!!!!!!!!!!!!!!
# OF DECISION VARIABLE SET CHANGES :    4


Decision var set #    1
      4.0000
      0.0000
      0.6667
      6.6667
# of occurances of this basis:   1000
% of overall occurance is: 100.000
AVE. OPTIMUM IS:   210.3186
Bias opt est (Ave Opt-True Opt):     -5.6814
The population variance is:       306.0520
The maximum value is:        264.7609
The minimum value is:        158.1760

Decision var set #    2
      1.0000
     11.0000
      0.0000
      0.0000
# of occurances of this basis:   620
% of overall occurance is:  62.000
AVE. OPTIMUM IS:   220.3541
Bias opt est (Ave Opt-True Opt):      4.3540
The population variance is:       334.3401
The maximum value is:        267.6013
The minimum value is:        169.1727

Decision var set #    3
      2.0000
      8.0000
      0.0000
      2.0000
# of occurances of this basis:   999
% of overall occurance is:  99.900
AVE. OPTIMUM IS:   220.8882
Bias opt est (Ave Opt-True Opt):      4.8881
The population variance is:       442.1299
The maximum value is:        276.3776
The minimum value is:        161.5952
```

```
Decision var set #   4
      8.0000
      0.0000
      2.0000
      0.0000
# of occurances of this basis:   19
% of overall occurance is:   1.900
AVE. OPTIMUM IS:   208.8170
Bias opt est (Ave Opt-True Opt):    -7.1830
The population variance is:      380.0007
The maximum value is:      260.9492
The minimum value is:      171.0470


!!!!!!!!!!!!!!!!!!! THE EXTREME POINT !!!!!!!!!!!!!!!!!!!
NUMBER OF EXT POINTS VISITED IS:   5

Extreme Point #   1
      4.0000
      0.0000
      0.6667
      6.6667
# OF EXT PT VISITS ARE:   1000
% OF OVERALL IS: 100.000
AVE. OPTIMUM IS:   210.3186
Bias opt est (Ave Opt-True Opt):    -5.6814

TRUE Z* WITH TRUE C IS:   215.3333
TRUE BIAS (Z* - Z optimal):    -0.6667
Difference between expected optimal and true    5.0147
The population variance is:      306.0520
The maximum value is:      264.7609
The minimum value is:      158.1760

Extreme Point #   2
      1.0000
     11.0000
      0.0000
      0.0000
# OF EXT PT VISITS ARE:   558
% OF OVERALL IS:  55.800
AVE. OPTIMUM IS:   220.3984
Bias opt est (Ave Opt-True Opt):    4.3984

TRUE Z* WITH TRUE C IS:   212.0000
TRUE BIAS (Z* - Z optimal):    -4.0000
Difference between expected optimal and true   -8.3984
The population variance is:      336.2216
The maximum value is:      267.6013
The minimum value is:      169.1727

Extreme Point #   3
      2.0000
```

```
      8.0000
      0.0000
      2.0000
# OF EXT PT VISITS ARE:   999
% OF OVERALL IS:  99.900
AVE. OPTIMUM IS:   220.8882
Bias opt est (Ave Opt-True Opt):     4.8881

TRUE Z* WITH TRUE C IS:   216.0000
TRUE BIAS (Z* - Z optimal):     0.0000
Difference between expected optimal and true   -4.8881
The population variance is:      442.1299
The maximum value is:      276.3776
The minimum value is:      161.5952

Extreme Point #   4
      2.0000
     10.0000
      0.0000
      0.0000
# OF EXT PT VISITS ARE:   62
% OF OVERALL IS:   6.200
AVE. OPTIMUM IS:   219.9551
Bias opt est (Ave Opt-True Opt):     3.9550

TRUE Z* WITH TRUE C IS:   166.0000
TRUE BIAS (Z* - Z optimal):   -50.0000
Difference between expected optimal and true  -53.9551
The population variance is:      317.2269
The maximum value is:      257.6813
The minimum value is:      174.5772

Extreme Point #   5
      8.0000
      0.0000
      2.0000
      0.0000
# OF EXT PT VISITS ARE:   19
% OF OVERALL IS:   1.900
AVE. OPTIMUM IS:   208.8170
Bias opt est (Ave Opt-True Opt):    -7.1830

TRUE Z* WITH TRUE C IS:    10.0000
TRUE BIAS (Z* - Z optimal):  -206.0000
Difference between expected optimal and true -198.8170
The population variance is:      380.0007
The maximum value is:      260.9492
The minimum value is:      171.0470


!!!!!!!!!!!!!!!!!!!!! OVERALL RESULTS !!!!!!!!!!!!!!!!!!!!!!

Overall Mean Optimum is:     6.1288
Overall Bias (Ave Opt-True Opt):  -209.8712
```

```
The overall sample variance is:     1298.7844
The overall maximum value is:      276.3776
The overall minimum value is:        0.0000


AVE OPTIMAL PER OBJECTIVE FUNCTION
Mean of Mean Opt per obj funct:     3.8361
The sample var(mean opt):        0.8616
The maximum value is:        8.1238
The minimum value is:        2.3285
```

```
NOISE MULTIPLIER (SD) FOR NORMAL NOISE:     13.0000
STANDARD ERROR IS :    3.25000
This is a screening run
   Using a Design to sample
# Times failed to sample true extreme pt  5
Standard deviation mult set =    1.50000    2.75000
4.00000
% failures overall:  0.500
Number of Objective Function Samples:   1000
Number of Runs per Obj Function:   149
Total Number of Points Tested:   149000
The True Objective Function:
const   10.00000+    15.0000    17.0000    18.0000
20.0000
Sample Generated Objective Function
    11.8447    15.9522    17.5825    18.9583    16.7721
```

Constraint Matrix

	1	2	3	4
1	1	1	2	1
2	2	1	-1	1
3	-1	1	1	2

```
The RHS is:     12.0000    14.0000   10.00000
*True Optimal Answer:      216.000
*True optimal Extreme Point:
    2.00000    8.00000  0.    2.00000
```

Design Matrix

	1	2	3	4	5
1	1	-1	-1	-1	-1
2	1	1	-1	-1	-1
3	1	-1	1	-1	-1
4	1	1	1	-1	-1
5	1	-1	-1	1	-1
6	1	1	-1	1	-1
7	1	-1	1	1	-1
8	1	1	1	1	-1
9	1	-1	-1	-1	1
10	1	1	-1	-1	1
11	1	-1	1	-1	1
12	1	1	1	-1	1
13	1	-1	-1	1	1
14	1	1	-1	1	1
15	1	-1	1	1	1
16	1	1	1	1	1

```
Sample response varaible Y:
   -45.8560    -25.5360    -27.8729     16.8729    -22.9366
2.03671     12.6677
    51.2050    -26.2992     4.34243    -8.73610     56.1767
25.7303     42.6163
    60.4428     74.6617
```

Sample Variance-Covariance Matrix

	1	2	3	4
1	7.179	0.000	0.000	0.000
2	0.000	7.179	0.000	0.000
3	0.000	0.000	7.179	0.000
4	0.000	0.000	0.000	7.179

Sample Cholesky Factorization Matrix

	1	2	3	4
1	2.679	0.000	0.000	0.000
2		2.679	0.000	0.000
3			2.679	0.000
4				2.679

SAMPLING DESING (BOX-BEHNKEN)

	1	2	3	4
1	-1.0	-0.5	0.0	0.0
2	1.0	-0.5	0.0	0.0
3	-1.0	0.5	0.0	0.0
4	1.0	0.5	0.0	0.0
5	0.0	0.0	-1.0	-0.5
6	0.0	0.0	1.0	-0.5
7	0.0	0.0	-1.0	0.5
8	0.0	0.0	1.0	0.5
9	-1.0	0.0	0.0	-0.5
10	1.0	0.0	0.0	-0.5
11	-1.0	0.0	0.0	0.5
12	1.0	0.0	0.0	0.5
13	0.0	-1.0	-0.5	0.0
14	0.0	1.0	-0.5	0.0
15	0.0	-1.0	0.5	0.0
16	0.0	1.0	0.5	0.0
17	-1.0	0.0	-0.5	0.0
18	1.0	0.0	-0.5	0.0
19	-1.0	0.0	0.5	0.0
20	1.0	0.0	0.5	0.0
21	0.0	-1.0	0.0	-0.5
22	0.0	1.0	0.0	-0.5
23	0.0	-1.0	0.0	0.5
24	0.0	1.0	0.0	0.5
25	0.0	0.0	0.0	0.0
26	-0.5	-1.0	0.0	0.0
27	0.5	-1.0	0.0	0.0
28	-0.5	1.0	0.0	0.0
29	0.5	1.0	0.0	0.0
30	0.0	0.0	-0.5	-1.0
31	0.0	0.0	0.5	-1.0
32	0.0	0.0	-0.5	1.0
33	0.0	0.0	0.5	1.0
34	-0.5	0.0	0.0	-1.0
35	0.5	0.0	0.0	-1.0
36	-0.5	0.0	0.0	1.0
37	0.5	0.0	0.0	1.0
38	0.0	-0.5	-1.0	0.0
39	0.0	0.5	-1.0	0.0

40	0.0	-0.5	1.0	0.0
41	0.0	0.5	1.0	0.0
42	-0.5	0.0	-1.0	0.0
43	0.5	0.0	-1.0	0.0
44	-0.5	0.0	1.0	0.0
45	0.5	0.0	1.0	0.0
46	0.0	-0.5	0.0	-1.0
47	0.0	0.5	0.0	-1.0
48	0.0	-0.5	0.0	1.0
49	0.0	0.5	0.0	1.0
50	0.0	0.0	0.0	0.0

```
!!!!!!!!!!!!!!!!!!! DECISION VARIABLES !!!!!!!!!!!!!!!!!!
# OF DECISION VARIABLE SET CHANGES :   6


Decision var set #   1
      4.0000
      0.0000
      0.6667
      6.6667
# of occurances of this basis:   999
% of overall occurance is:  99.900
AVE. OPTIMUM IS:   208.5842
Bias opt est (Ave Opt-True Opt):     -7.4158
The population variance is:      651.1973
The maximum value is:      288.6066
The minimum value is:      137.1993

Decision var set #   2
      2.0000
      8.0000
      0.0000
      2.0000
# of occurances of this basis:   995
% of overall occurance is:  99.500
AVE. OPTIMUM IS:   222.7262
Bias opt est (Ave Opt-True Opt):     6.7262
The population variance is:      841.8345
The maximum value is:      310.9501
The minimum value is:      145.5584

Decision var set #   3
      1.0000
     11.0000
      0.0000
      0.0000
# of occurances of this basis:   806
% of overall occurance is:  80.600
AVE. OPTIMUM IS:   223.1638
Bias opt est (Ave Opt-True Opt):     7.1638
The population variance is:      721.3096
The maximum value is:      308.3556
The minimum value is:      150.9656
```

```
Decision var set #   4
      8.0000
      0.0000
      2.0000
      0.0000
# of occurances of this basis:   103
% of overall occurance is:  10.300
AVE. OPTIMUM IS:   225 2866
Bias opt est (Ave Opt-True Opt):     9.2865
The population variance is:      760.8097
The maximum value is:      283.2927
The minimum value is:      151.0016

Decision var set #   5
      0.0000
      0.0000
      4.6667
      2.6667
# of occurances of this basis:   7
% of overall occurance is:   0.700
AVE. OPTIMUM IS:   168.4372
Bias opt est (Ave Opt-True Opt):   -47.5629
The population variance is:      661.1362
The maximum value is:      219.5729
The minimum value is:      139.5171

Decision var set #   6
      0.0000
      8.0000
      2.0000
      0.0000
# of occurances of this basis:   5
% of overall occurance is:   0.500
AVE. OPTIMUM IS:   163.4464
Bias opt est (Ave Opt-True Opt):   -52.5536
The population variance is:      165.5492
The maximum value is:      187.7662
The minimum value is:      149.7473


!!!!!!!!!!!!!!!!!!!!  THE EXTREME POINT !!!!!!!!!!!!!!!!!!!!
NUMBER OF EXT POINTS VISITED IS:   7

Extreme Point #   1
      4.0000
      0.0000
      0.6667
      6.6667
# OF EXT PT VISITS ARE:   999
% OF OVERALL IS:  99.900
AVE. OPTIMUM IS:   208.5842
Bias opt est (Ave Opt-True Opt):    -7.4158
```

```
TRUE Z* WITH TRUE C IS:   215.3333
TRUE BIAS (Z* - Z optimal):     -0.6667
Difference between expected optimal and true     6.7491
The population variance is:       651.1973
The maximum value is:      288.6066
The minimum value is:      137.1993

Extreme Point #    2
      2.0000
      8.0000
      0.0000
      2.0000
# OF EXT PT VISITS ARE:   995
% OF OVERALL IS:  99.500
AVE. OPTIMUM IS:   222.7262
Bias opt est (Ave Opt-True Opt):     6.7262

TRUE Z* WITH TRUE C IS:   216.0000
TRUE BIAS (Z* - Z optimal):     0.0000
Difference between expected optimal and true    -6.7262
The population variance is:       841.8345
The maximum value is:      310.9501
The minimum value is:      145.5584

Extreme Point #    3
      1.0000
     11.0000
      0.0000
      0.0000
# OF EXT PT VISITS ARE:   661
% OF OVERALL IS:  66.100
AVE. OPTIMUM IS:   221.9296
Bias opt est (Ave Opt-True Opt):     5.9295

TRUE Z* WITH TRUE C IS:   212.0000
TRUE BIAS (Z* - Z optimal):     -4.0000
Difference between expected optimal and true    -9.9296
The population variance is:       700.3993
The maximum value is:      297.5756
The minimum value is:      153.1106

Extreme Point #    4
      2.0000
     10.0000
      0.0000
      0.0000
# OF EXT PT VISITS ARE:   145
% OF OVERALL IS:  14.500
AVE. OPTIMUM IS:   228.7902
Bias opt est (Ave Opt-True Opt):    12.7901

TRUE Z* WITH TRUE C IS:   166.0000
TRUE BIAS (Z* - Z optimal):   -50.0000
Difference between expected optimal and true   -62.7902
```

```
The population variance is:      778.0309
The maximum value is:      308.3556
The minimum value is:      150.9656

Extreme Point #   5
      8.0000
      0.0000
      2.0000
      0.0000
# OF EXT PT VISITS ARE:   103
% OF OVERALL IS:  10.300
AVE. OPTIMUM IS:   225.2866
Bias opt est (Ave Opt-True Opt):     9.2865

TRUE Z* WITH TRUE C IS:   147.3333
TRUE BIAS (Z* - Z optimal):   -68.6667
Difference between expected optimal and true  -77.9532
The population variance is:      760.8097
The maximum value is:      283.2927
The minimum value is:      151.0016

Extreme Point #   6
      0.0000
      0.0000
      4.6667
      2.6667
# OF EXT PT VISITS ARE:   7
% OF OVERALL IS:   0.700
AVE. OPTIMUM IS:   168.4372
Bias opt est (Ave Opt-True Opt):   -47.5629

TRUE Z* WITH TRUE C IS:   182.0000
TRUE BIAS (Z* - Z optimal):   -34.0000
Difference between expected optimal and true   13.5628
The population variance is:      661.1362
The maximum value is:      219.5729
The minimum value is:      139.5171

Extreme Point #   7
      0.0000
      8.0000
      2.0000
      0.0000
# OF EXT PT VISITS ARE:   5
% OF OVERALL IS:   0.500
AVE. OPTIMUM IS:   163.4464
Bias opt est (Ave Opt-True Opt):   -52.5536

TRUE Z* WITH TRUE C IS:    10.0000
TRUE BIAS (Z* - Z optimal):  -206.0000
Difference between expected optimal and true -153.4464
The population variance is:      165.5492
The maximum value is:      187.7662
The minimum value is:      149.7473
```

```
!!!!!!!!!!!!!!!!!!!!! OVERALL RESULTS !!!!!!!!!!!!!!!!!!!!!!

Overall Mean Optimum is:      7.9645
Overall Bias (Ave Opt-True Opt):  -208.0355
The overall sample variance is:     1695.8068
The overall maximum value is:      310.9501
The overall minimum value is:        0.0000


AVE OPTIMAL PER OBJECTIVE FUNCTION
Mean of Mean Opt per obj funct:      4.2621
The sample var(mean opt):        1.5189
The maximum value is:         9.1224
The minimum value is:         2.0629
```

NOISE MULTIPLIER (SD) FOR NORMAL NOISE: 17.0000
STANDARD ERROR IS : 4.25000
This is a screening run
Using a Design to sample
Times failed to sample true extreme pt 19
Standard deviation mult set = 1.50000 2.75000
4.00000
% failures overall: 1.900
Number of Objective Function Samples: 1000
Number of Runs per Obj Function: 149
Total Number of Points Tested: 149000
The True Objective Function:
const 10.00000+ 15.0000 17.0000 18.0000
20.0000
Sample Generated Objective Function
8.30373 10.6681 21.0220 19.5529 19.5255

Constraint Matrix

	1	2	3	4
1	1	1	2	1
2	2	1	-1	1
3	-1	1	1	2

The RHS is: 12.0000 14.0000 10.00000
*True Optimal Answer: 216.000
*True optimal Extreme Point:
2.00000 8.00000 0. 2.00000

Design Matrix

	1	2	3	4	5
1	1	-1	-1	-1	-1
2	1	1	-1	-1	-1
3	1	-1	1	-1	-1
4	1	1	1	-1	-1
5	1	-1	-1	1	-1
6	1	1	-1	1	-1
7	1	-1	1	1	-1
8	1	1	1	1	-1
9	1	-1	-1	-1	1
10	1	1	-1	-1	1
11	1	-1	1	-1	1
12	1	1	1	-1	1
13	1	-1	-1	1	1
14	1	1	-1	1	1
15	1	-1	1	1	1
16	1	1	1	1	1

Sample response varaible Y:
-54.3485 -48.7402 -26.4912 16.6902 -35.1069
4.32857 2.29300
51.6007 -26.1163 -5.40654 16.1785 38.2412
32.9653 30.6787
71.7110 64.3823

Sample Variance-Covariance Matrix

	1	2	3	4
1	10.04	0.00	0.00	0.00
2	0.00	10.04	0.00	0.00
3	0.00	0.00	10.04	0.00
4	0.00	0.00	0.00	10.04

Sample Cholesky Factorization Matrix

	1	2	3	4
1	3.169	0.000	0.000	0.000
2		3.169	0.000	0.000
3			3.169	0.000
4				3.169

SAMPLING DESING (BOX-BEHNKEN)

	1	2	3	4
1	-1.0	-0.5	0.0	0.0
2	1.0	-0.5	0.0	0.0
3	-1.0	0.5	0.0	0.0
4	1.0	0.5	0.0	0.0
5	0.0	0.0	-1.0	-0.5
6	0.0	0.0	1.0	-0.5
7	0.0	0.0	-1.0	0.5
8	0.0	0.0	1.0	0.5
9	-1.0	0.0	0.0	-0.5
10	1.0	0.0	0.0	-0.5
11	-1.0	0.0	0.0	0.5
12	1.0	0.0	0.0	0.5
13	0.0	-1.0	-0.5	0.0
14	0.0	1.0	-0.5	0.0
15	0.0	-1.0	0.5	0.0
16	0.0	1.0	0.5	0.0
17	-1.0	0.0	-0.5	0.0
18	1.0	0.0	-0.5	0.0
19	-1.0	0.0	0.5	0.0
20	1.0	0.0	0.5	0.0
21	0.0	-1.0	0.0	-0.5
22	0.0	1.0	0.0	-0.5
23	0.0	-1.0	0.0	0.5
24	0.0	1.0	0.0	0.5
25	0.0	0.0	0.0	0.0
26	-0.5	-1.0	0.0	0.0
27	0.5	-1.0	0.0	0.0
28	-0.5	1.0	0.0	0.0
29	0.5	1.0	0.0	0.0
30	0.0	0.0	-0.5	-1.0
31	0.0	0.0	0.5	-1.0
32	0.0	0.0	-0.5	1.0
33	0.0	0.0	0.5	1.0
34	-0.5	0.0	0.0	-1.0
35	0.5	0.0	0.0	-1.0
36	-0.5	0.0	0.0	1.0
37	0.5	0.0	0.0	1.0
38	0.0	-0.5	-1.0	0.0
39	0.0	0.5	-1.0	0.0

40	0.0	-0.5	1.0	0.0
41	0.0	0.5	1.0	0.0
42	-0.5	0.0	-1.0	0.0
43	0.5	0.0	-1.0	0.0
44	-0.5	0.0	1.0	0.0
45	0.5	0.0	1.0	0.0
46	0.0	-0.5	0.0	-1.0
47	0.0	0.5	0.0	-1.0
48	0.0	-0.5	0.0	1.0
49	0.0	0.5	0.0	1.0
50	0.0	0.0	0.0	0.0

```
!!!!!!!!!!!!!!!!!!! DECISION VARIABLES !!!!!!!!!!!!!!!!!!
# OF DECISION VARIABLE SET CHANGES :    7


Decision var set #   1
      4.0000
      0.0000
      0.6667
      6.6667
# of occurances of this basis:   1001
% of overall occurance is: 100.100
AVE. OPTIMUM IS:   207.5924
Bias opt est (Ave Opt-True Opt):    -8.4077
The population variance is:      973.1924
The maximum value is:      304.9121
The minimum value is:      103.5766

Decision var set #   2
      1.0000
     11.0000
      0.0000
      0.0000
# of occurances of this basis:   912
% of overall occurance is:  91.200
AVE. OPTIMUM IS:   225.7134
Bias opt est (Ave Opt-True Opt):     9.7133
The population variance is:     1225.7222
The maximum value is:      336.5096
The minimum value is:      129.2516

Decision var set #   3
      2.0000
      8.0000
      0.0000
      2.0000
# of occurances of this basis:   981
% of overall occurance is:  98.100
AVE. OPTIMUM IS:   227.2527
Bias opt est (Ave Opt-True Opt):    11.2526
The population variance is:     1432.8307
The maximum value is:      371.2051
The minimum value is:      120.2687
```

```
Decision var set #   4
      8.0000
      0.0000
      2.0000
      0.0000
# of occurances of this basis:   188
% of overall occurance is:  18.800
AVE. OPTIMUM IS:   232.0063
Bias opt est (Ave Opt-True Opt):    16.0063
The population variance is:     1289.7496
The maximum value is:       319.6638
The minimum value is:       122.2305

Decision var set #   5
      0.0000
      8.0000
      2.0000
      0.0000
# of occurances of this basis:   51
% of overall occurance is:   5.100
AVE. OPTIMUM IS:   174.5180
Bias opt est (Ave Opt-True Opt):   -41.4821
The population variance is:      595.1209
The maximum value is:       222.3652
The minimum value is:       113.8435

Decision var set #   6
      0.0000
      0.0000
      4.6667
      2.6667
# of occurances of this basis:   47
% of overall occurance is:   4.700
AVE. OPTIMUM IS:   164.8198
Bias opt est (Ave Opt-True Opt):   -51.1802
The population variance is:      478.3920
The maximum value is:       223.9326
The minimum value is:       128.0441

Decision var set #   7
      0.0000
      0.0000
      6.0000
      0.0000
# of occurances of this basis:   19
% of overall occurance is:   1.900
AVE. OPTIMUM IS:   175.7839
Bias opt est (Ave Opt-True Opt):   -40.2161
The population variance is:     1132.7343
The maximum value is:       243.3237
The minimum value is:       127.3116
```

```
!!!!!!!!!!!!!!!!!!!  THE EXTREME POINT !!!!!!!!!!!!!!!!!!!
NUMBER OF EXT POINTS VISITED IS:   8

Extreme Point #   1
      4.0000
      0.0000
      0.6667
      6.6667
# OF EXT PT VISITS ARE:   1001
% OF OVERALL IS: 100.100
AVE. OPTIMUM IS:   207.5924
Bias opt est (Ave Opt-True Opt):    -8.4077

TRUE Z* WITH TRUE C IS:   215.3333
TRUE BIAS (Z* - Z optimal):    -0.6667
Difference between expected optimal and true    7.7410
The population variance is:      973.1924
The maximum value is:      304.9121
The minimum value is:      103.5766

Extreme Point #   2
      1.0000
     11.0000
      0.0000
      0.0000
# OF EXT PT VISITS ARE:   736
% OF OVERALL IS:  73.600
AVE. OPTIMUM IS:   224.2276
Bias opt est (Ave Opt-True Opt):    8.2276

TRUE Z* WITH TRUE C IS:   212.0000
TRUE BIAS (Z* - Z optimal):    -4.0000
Difference between expected optimal and true  -12.2276
The population variance is:     1152.1857
The maximum value is:      314.2408
The minimum value is:      129.2516

Extreme Point #   3
      2.0000
     10.0000
      0.0000
      0.0000
# OF EXT PT VISITS ARE:   176
% OF OVERALL IS:  17.600
AVE. OPTIMUM IS:   231.9267
Bias opt est (Ave Opt-True Opt):    15.9266

TRUE Z* WITH TRUE C IS:   216.0000
TRUE BIAS (Z* - Z optimal):     0.0000
Difference between expected optimal and true  -15.9266
The population variance is:     1485.4053
The maximum value is:      336.5096
The minimum value is:      129.5463
```

```
Extreme Point #   4
      2.0000
      8.0000
      0.0000
      2.0000
# OF EXT PT VISITS ARE:   981
% OF OVERALL IS:  98.100
AVE. OPTIMUM IS:   227.2527
Bias opt est (Ave Opt-True Opt):    11.2526

TRUE Z* WITH TRUE C IS:   166.0000
TRUE BIAS (Z* - Z optimal):   -50.0000
Difference between expected optimal and true  -61.2527
The population variance is:     1432.8307
The maximum value is:      371.2051
The minimum value is:      120.2687

Extreme Point #   5
      8.0000
      0.0000
      2.0000
      0.0000
# OF EXT PT VISITS ARE:   188
% OF OVERALL IS:  18.800
AVE. OPTIMUM IS:   232.0063
Bias opt est (Ave Opt-True Opt):    16.0063

TRUE Z* WITH TRUE C IS:   182.0000
TRUE BIAS (Z* - Z optimal):   -34.0000
Difference between expected optimal and true  -50.0063
The population variance is:     1289.7496
The maximum value is:      319.6638
The minimum value is:      122.2305

Extreme Point #   6
      0.0000
      8.0000
      2.0000
      0.0000
# OF EXT PT VISITS ARE:   51
% OF OVERALL IS:   5.100
AVE. OPTIMUM IS:   174.5180
Bias opt est (Ave Opt-True Opt):   -41.4821

TRUE Z* WITH TRUE C IS:   147.3333
TRUE BIAS (Z* - Z optimal):   -68.6667
Difference between expected optimal and true  -27.1846
The population variance is:      595.1209
The maximum value is:      222.3652
The minimum value is:      113.8435

Extreme Point #   7
      0.0000
      0.0000
```

```
      4.6667
      2.6667
# OF EXT PT VISITS ARE:   47
% OF OVERALL IS:   4.700
AVE. OPTIMUM IS:   164.8198
Bias opt est (Ave Opt-True Opt):   -51.1802

TRUE Z* WITH TRUE C IS:   118.0000
TRUE BIAS (Z* - Z optimal):   -98.0000
Difference between expected optimal and true  -46.8198
The population variance is:      478.3920
The maximum value is:       223.9326
The minimum value is:       128.0441

Extreme Point #   8
      0.0000
      0.0000
      6.0000
      0.0000
# OF EXT PT VISITS ARE:   19
% OF OVERALL IS:   1.900
AVE. OPTIMUM IS:   175.7839
Bias opt est (Ave Opt-True Opt):   -40.2161

TRUE Z* WITH TRUE C IS:    10.0000
TRUE BIAS (Z* - Z optimal):  -206.0000
Difference between expected optimal and true -165.7839
The population variance is:     1132.7343
The maximum value is:       243.3237
The minimum value is:       127.3116


!!!!!!!!!!!!!!!!!!!! OVERALL RESULTS !!!!!!!!!!!!!!!!!!!!!!

Overall Mean Optimum is:      9.9972
Overall Bias (Ave Opt-True Opt):  -206.0028
The overall sample variance is:     2134.0967
The overall maximum value is:     371.2051
The overall minimum value is:       0.0000


AVE OPTIMAL PER OBJECTIVE FUNCTION
Mean of Mean Opt per obj funct:     4.6993
The sample var(mean opt):        2.0915
The maximum value is:      10.7572
The minimum value is:       1.6353
```

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Vita

First Lieutenant Robert Garrison Harvey was born on 11 August 1966 in Kansas City, Missouri. He graduated from Central Kitsap High School in Silverdale, Washington in 1984 and attended the U.S. Air Force Academy, graduating with a Bachelor of Science in Applied Mathematics in June 1988. Upon graduation, he received a regular commission in the USAF and served his first tour of duty with the Air Force Test and Evaluation Center at Vandenburg AFB, California. He was a reliability analyst monitoring and evaluating tests on the Peacekeeper missile, Peacekeeper Railgarrison system, and Small ICBM until entering the School of Engineering, Air Force Institute of Technology, in August 1990. After graduation he started a new tour of duty with the Command Analysis Group at HQ Military Airlift Command, Scott AFB, IL (618) 256-3119

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13. ABSTRACT
This research investigated an alternative to the traditional approaches of optimizing a stochastic response surface subject to constraints. This research investigated the bias in the expected value of the solution. A three step process is presented to evaluate stochastic response surfaces subject to constraints. Step 1 uses a traditional approach to estimate the response surface and a covariance matrix through regression. Step 2 samples the objective function of the linear program (i.e. the response surface) and identifies the extreme points visited. Step 3 presents a method to estimate the optimal extreme point and present that information to a decision maker.

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